

Deriving Models for keV sterile Neutrino Dark Matter with the Froggatt-Nielsen mechanism

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January 25, 2013

Abstract

Sterile neutrinos with a mass around the keV scale are an attractive particle physics candidate for Warm Dark Matter. Although many frameworks have been presented in which these neutrinos can fulfill all phenomenological constraints, there are hardly any models known that can explain such a peculiar mass pattern, one sterile neutrino at the keV scale and the other two considerably heavier, while at the same time being compatible with low-energy neutrino data. In this paper, we present models based on the Froggatt-Nielsen mechanism, which can give such an explanation. We explain how to assign Froggatt-Nielsen charges in a successful way, and we give a detailed discussion of all conditions to be fulfilled. It turns out that the typical arbitrariness of the charge assignments is greatly reduced when trying to carefully account for all constraints. We furthermore present analytical calculations of a few simplified models, while quasi-perfect models are found numerically.

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1 Introduction

One of the most intriguing problems in today’s particle physics and astrophysics is the identity of the mysterious Dark Matter [1] observed in the Universe [2]. It is known since a few years that computer simulations of small scale structure formation are in favour of so-called “warm” Dark Matter (WDM), with a mass of roughly 1–2 keV [3, 4], which is an intermediate case between the standard paradigm of “cold” (non-relativistic) and “hot” (relativistic) Dark Matter, the latter being strongly disfavored by structure formation. Furthermore, there are model-independent data analyses which also seem to point to the keV scale [5].

A very interesting framework providing a WDM candidate motivated by particle physics is the so-called ν MSM [6], which extends the Standard Model (SM) of particle physics by three right-handed (sterile) neutrinos, one of which has a mass at the keV scale, whereas the other two are considerably heavier. However, it turns out that, in the standard thermal production, the Dark Matter would be overproduced, which makes it necessary to rely on non-thermal production instead [7, 8, 9]. An alternative is provided by embedding the keV sterile neutrino in a gauge extension of the SM and correcting the abundance by allowing for sufficient entropy production in the decay of the two heavier sterile neutrinos, which has been exemplified in a Left-Right symmetric framework [10]. Note, however, that this last proposal required a seesaw type II situation in order not to be in conflict with the experimental and observational constraints. In addition, keV sterile neutrinos can also appear in the frameworks of the scotogenic/inert Higgs doublet model [11, 12] or of composite neutrinos [13].

The common feature of all these proposals is, however, that they do not yield an *explanation* of the required mass pattern of sterile neutrinos: They can be seen as very useful frameworks providing all tools to make concrete predictions, and the corresponding parameters can assume values leading to full agreement with data. Nevertheless, in the view of model building, they only assume the correct mass pattern to be present.

Up to now, to our knowledge, only two classes models have existed which could yield an explanation of the sterile neutrino mass pattern. The key point is to achieve a strong mass splitting in the right-handed neutrino sector, as only the lightest sterile particle should have a mass in the keV range, while the other two must be considerably heavier [10]. This leads to different schemes for shifting a certain initial mass spectrum, two of which are depicted in Fig. 1. The first class of models is based on a flavour symmetry which forces one sterile neutrino to be strictly massless. This idea has first been discussed in Ref. [14] for a specifically defined lepton number symmetry, and has recently been investigated in the context of a $L_e - L_\mu - L_\tau$ flavour symmetry [15]. Both symmetries can explain such a peculiar pattern of right-handed neutrino masses easily: The trick is to make use of the different scales of symmetry breaking and symmetry preserving terms to generate the necessary hierarchy. Soft breaking of the symmetry will lift up the mass of one sterile neutrino, which would have been strictly massless in the limit of restored symmetry, to the keV scale, see left panel of Fig. 1 for the $L_e - L_\mu - L_\tau$ case. The breaking will furthermore break up the predicted exact mass degeneracy in the light neutrino sector,

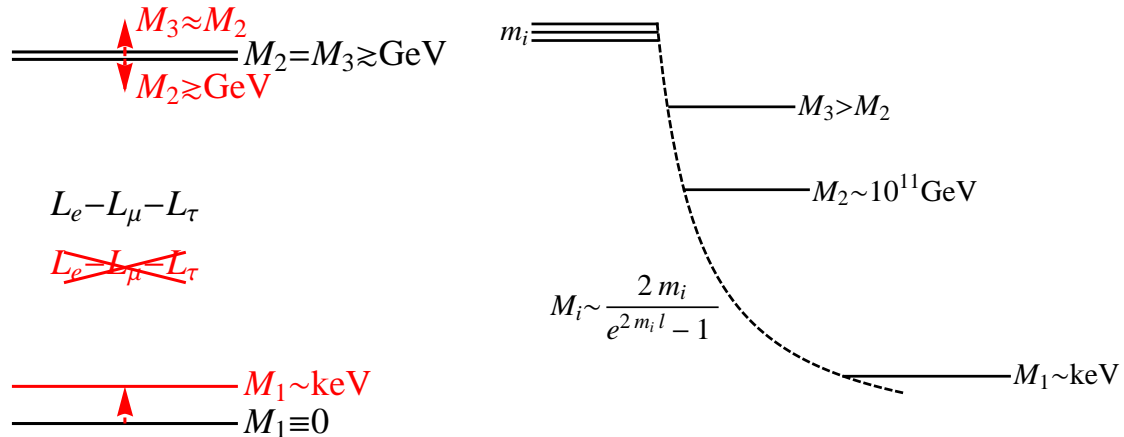


Figure 1: The mass shifting schemes of the two classes of models that had already existed: The left panel displays the shifting due to soft breaking of a global $L_e - L_\mu - L_\tau$ flavour symmetry, while the right one shows the exponential suppression in a Randall-Sundrum framework.

which would otherwise contradict the data. The second real model [16] exploited the exponential factor in a Randall-Sundrum [17] like framework to obtain a large mass splitting from very moderately tuned parameters, cf. right panel of Fig. 1. This model has the advantages of naturally explaining the existence of one sterile neutrino at the keV scale, while the other two could have masses of around 10^{11} GeV or even heavier, and of at the same time insuring that the seesaw mechanism [18, 19, 20, 21, 22] works, in spite of the existence of a relatively light sterile neutrino. On the other hand, this model involves the assumption of a suitable UV-brane, which is intrinsically hard to probe.

In this paper, we want to present a third alternative to explain the existence of a keV scale sterile neutrino, by making use of the Froggatt-Nielsen (FN) mechanism [23]. This mechanism is well-known to be capable of explaining very strong mass hierarchies, e.g. in the quark sector [24, 25, 26], which is just what is needed for having one sterile neutrino at the keV scale whereas the other two are heavier by at least six orders of magnitude. Our goal is to find the *minimal* assignments needed to explain the pattern in the sterile neutrino sector, while being in full agreement with the rest of the lepton data. The key point of the FN mechanism is that at most one fermion generation (namely the third one, if at all) obtains a mass by the standard Higgs mechanism, or by the existence of a direct mass term for gauge singlets. The other generations, however, will only receive higher-order contributions from multiple seesaw-like diagrams, which leads to the typical cascade or waterfall structures [27]. This is enforced by having an additional Abelian $U(1)_{\text{FN}}$ flavour symmetry under which the first, second, and third generations have charges with decreasing absolute values. Furthermore, such a framework requires a heavy sector of new fermions that is usually not specified in details, as well as so-called flavon fields, which are SM singlets but charged under the $U(1)_{\text{FN}}$. These fields obtain vacuum expectation values (VEVs) in order to break the symmetry in a phenomenologically suitable way [28].

Integrating out the heavy fields finally leads to suppression factors by some power of a small parameter λ , which is usually taken to be of the order of the Cabibbo angle [29].

Note that, shortly before this work was completed, Ref. [30] appeared, where a Froggatt-Nielsen mechanism, together with a Z_3 symmetry, is used in the context of an A_4 flavour symmetry to generate a hierarchy for the charged lepton masses and to regulate the mass of a sterile neutrino to be around the eV-scale. The authors also comment on the possibility of exploiting this model for an explanation of a sterile neutrino with a keV mass, however, without investigating this route explicitly. In their case, however, the Froggatt-Nielsen mechanism is merely a small addendum to the existing A_4 model, whereas we consider the mechanism as starting point to develop fully working models.

The structure of this paper is the following: In Sec. 2 the main features of the Froggatt-Nielsen mechanism are summarized. The possible choices for the FN charges of the right-handed neutrinos are analyzed in Sec. 3. After that, in Sec. 4, we give a detailed discussion about all requirements to be fulfilled, where we will also explain why certain frameworks are incompatible with the FN mechanism when aiming at a description of keV mass sterile neutrinos. In addition, we will also discuss why certain constraints that might be problematic for a higher right-handed neutrino mass scale are practically of no relevance to our case. Finally, in Sec. 5, we present analytical calculations of some promising scenarios, as well as fully working numerical models, before concluding in Sec. 6. Explicit expressions for the diagonalization matrices used can be found in Appendix A.

2 The Froggatt-Nielsen mechanism

The Froggatt-Nielsen (FN) mechanism [23] is maybe one of the best possibilities to explain strong hierarchies between fermion masses. Furthermore, there exist successful explicit applications of this mechanism to the neutrino sector, see, e.g., Refs. [26, 31], and Refs. [32, 33, 34, 35] for a systematic parameter space scan.

We would like to use the FN mechanism to explain a hierarchical spectrum (HS) in the sterile neutrino sector of the following type: $M_1 \simeq \mathcal{O}(\text{keV})$ and $M_2, M_3 \gtrsim \mathcal{O}(\text{GeV})$. Denoting the FN flavon field by Θ , and the cut-off scale at which the heavy sector of the theory is integrated out by Λ , we can write the Lagrangian as:

$$\begin{aligned} \mathcal{L}_{\text{leptons}} = & -Y_e^{ij} \overline{e_{iR}} H L_{jL} \left(\frac{\Theta}{\Lambda} \right)^{k_i+f_j} + h.c. - Y_D^{ij} \overline{N_{iR}} \tilde{H} L_{jL} \left(\frac{\Theta}{\Lambda} \right)^{g_i+f_j} + h.c. \\ & - \frac{1}{2} \overline{N_{iR}} \tilde{M}_R^{ij} (N_{jR})^C \left(\frac{\Theta}{\Lambda} \right)^{g_i+g_j} + h.c. - \frac{1}{2} Y_L^{ij} \overline{(L_{iL})^C} (i\sigma_2 \Delta) L_{jL} \left(\frac{\Theta}{\Lambda} \right)^{f_i+f_j} + h.c., \end{aligned} \quad (1)$$

where H is the Standard Model Higgs and $\tilde{H} = i\sigma_2 H^*$ is its charge conjugate (see Tab. 1 for the definition of all the other fields). The FN flavon field can acquire a VEV $\langle \Theta \rangle$, with $\lambda = \frac{\langle \Theta \rangle}{\Lambda}$ being a small quantity of the order of the Cabibbo angle: $\lambda \simeq 0.22$ [29].

Note that, in principle, FN charges could be positive or negative, which would correspond to integrating out heavy particles or anti-particles, respectively. This would again lead to

Field	L_{iL}	$\overline{e_{iR}}$	$\overline{N_{iR}}$	H	Δ	Θ
$SU(2)_L$	<u>2</u>	<u>1</u>	<u>1</u>	<u>2</u>	<u>3</u>	<u>1</u>
$U(1)_{\text{FN}}$	f_i	k_i	g_i	0	0	-1

Table 1: $SU(2)$ and family charges, with $i = 1, 2, 3$.

suppression factors due to having a higher mass scale involved. Accordingly, the decisive quantity for the suppression is actually the absolute value of the sum of FN charges. After the flavon field acquires a VEV and after electroweak symmetry breaking, the charged leptons and the Dirac neutrino mass matrices are given by

$$M_e = v \begin{pmatrix} Y_e^{11} \lambda^{|k_1+f_1|} & Y_e^{12} \lambda^{|k_1+f_2|} & Y_e^{13} \lambda^{|k_1+f_3|} \\ Y_e^{21} \lambda^{|k_2+f_1|} & Y_e^{22} \lambda^{|k_2+f_2|} & Y_e^{23} \lambda^{|k_2+f_3|} \\ Y_e^{31} \lambda^{|k_3+f_1|} & Y_e^{32} \lambda^{|k_3+f_2|} & Y_e^{33} \lambda^{|k_3+f_3|} \end{pmatrix}, \quad (2)$$

$$m_D = v \begin{pmatrix} Y_D^{11} \lambda^{|g_1+f_1|} & Y_D^{12} \lambda^{|g_1+f_2|} & Y_D^{13} \lambda^{|g_1+f_3|} \\ Y_D^{21} \lambda^{|g_2+f_1|} & Y_D^{22} \lambda^{|g_2+f_2|} & Y_D^{23} \lambda^{|g_2+f_3|} \\ Y_D^{31} \lambda^{|g_3+f_1|} & Y_D^{32} \lambda^{|g_3+f_2|} & Y_D^{33} \lambda^{|g_3+f_3|} \end{pmatrix}, \quad (3)$$

where v is the VEV of the Higgs doublet and f_i , k_i , and g_i are the FN charges of the left-handed lepton doublets, the right-handed charged, and the right-handed neutrinos, respectively, see Tab. 1.

Now let us have a look at the right-handed neutrino sector. Since the corresponding mass matrix M_R has to be symmetric, it will have the following structure:

$$M_R = \begin{pmatrix} \tilde{M}_R^{11} \lambda^{|2g_1|} & \tilde{M}_R^{12} \lambda^{|g_1+g_2|} & \tilde{M}_R^{13} \lambda^{|g_1+g_3|} \\ \bullet & \tilde{M}_R^{22} \lambda^{|2g_2|} & \tilde{M}_R^{23} \lambda^{|g_2+g_3|} \\ \bullet & \bullet & \tilde{M}_R^{33} \lambda^{|2g_3|} \end{pmatrix}. \quad (4)$$

Applying the type I seesaw formula then leads to the following structure:

$$m_\nu^I = -m_D^T M_R^{-1} m_D = \begin{pmatrix} a_1 \lambda^{|2f_1|} & b_1 \lambda^{|f_1+f_2|} & c_1 \lambda^{|f_1+f_3|} \\ \bullet & d_1 \lambda^{|2f_2|} & e_1 \lambda^{|f_2+f_3|} \\ \bullet & \bullet & f_1 \lambda^{|2f_3|} \end{pmatrix}, \quad (5)$$

where the parameters $a_1, b_1, c_1, d_1, e_1, f_1$ depend on the parameters present in the mass matrices m_D and M_R . In the case of a type II seesaw scenario, one needs the left-handed Majorana neutrino mass in addition,

$$m_L = v_\Delta \begin{pmatrix} Y_L^{11} \lambda^{|2f_1|} & Y_L^{12} \lambda^{|f_1+f_2|} & Y_L^{13} \lambda^{|f_1+f_3|} \\ \bullet & Y_L^{22} \lambda^{|2f_2|} & Y_L^{23} \lambda^{|f_2+f_3|} \\ \bullet & \bullet & Y_L^{33} \lambda^{|2f_3|} \end{pmatrix}, \quad (6)$$

with v_Δ being the VEV of the triplet Δ . The type II seesaw neutrino mass matrix will be given by

$$m_\nu^{II} = m_L - m_D^T M_R^{-1} m_D = m_L + m_\nu^I = \begin{pmatrix} a_2 \lambda^{|2f_1|} & b_2 \lambda^{|f_1+f_2|} & c_2 \lambda^{|f_1+f_3|} \\ \bullet & d_2 \lambda^{|2f_2|} & e_2 \lambda^{|f_2+f_3|} \\ \bullet & \bullet & f_2 \lambda^{|2f_3|} \end{pmatrix}, \quad (7)$$

where the parameters $a_2, b_2, c_2, d_2, e_2, f_2$ depend on the parameters present in the mass matrices m_L , m_D , and M_R .

Note that Eqs. (5) and (7) proof that the seesaw mechanism is, in this framework, not spoiled by the presence of a keV neutrino: Any global $U(1)$ charge would cancel out in the seesaw formula, and so do the $U(1)_{\text{FN}}$ charges of the right-handed fermions. These charges are, however, the only connection to the keV scale, since they lower the natural scale M_0 of the right-handed neutrino mass by introducing certain powers of λ . Accordingly, in the seesaw formula, one is always guaranteed to divide by a relatively large value M_0 , thereby saving the seesaw mechanism. In fact, since the FN charges of the left-handed lepton doublets are still present in the seesaw formula, the suppression mechanism for the light neutrino masses might even be amplified.

Furthermore one can see from Eqs. (5) and (7) that the structure of the neutrino mass matrices is precisely the same for type I and type II seesaw scenarios, which is another consequence of the right-handed neutrino charges canceling out. For this reason, we will not consider type II seesaw scenarios any further in this paper, since the results for such scenarios can be trivially recovered from the seesaw type I results, by the simple transformations $a_1 \rightarrow a_2$, $b_1 \rightarrow b_2$, and so on. We will, however, mention differences or additional features that would appear in the type II seesaw case, where appropriate.

Before starting to apply the FN mechanism to explain keV sterile neutrinos, we also want to stress a potential problem of this method: The Froggatt-Nielsen mechanism intrinsically involves a high energy sector that is not specified further. This is a problem not only of this particular mechanism, but of practically all flavour symmetries that must be broken in a phenomenologically acceptable way. Such a breaking always involves a scalar (flavon) sector, which is assumed to have suitable properties. In addition, the FN mechanism also involves heavy fermions, which are integrated out to lead to the suppression factors in Eq. (1). Such high energy sectors, although of no practical relevance for low energy experiments, could potentially become important for very high energies, and hence in particular in the early Universe. This problem, however, is strongly model-dependent, and far beyond the scope of a conceptual paper like the one presented here.

We nevertheless want to point out that the additional interactions generated by this high energy sector could, e.g., cause the keV sterile neutrinos to be in thermal equilibrium at early times. Although certain production mechanisms of keV sterile neutrino DM require the sterile neutrinos to never enter thermal equilibrium (see, e.g., Refs. [6, 7, 8, 9]), there is also the alternative possibility to first produce them thermally and then dilute their abundance by sufficient entropy production [10]. The lesson to learn is that one should be careful when applying the FN mechanism to certain scenarios: We will discuss many accompanying problems from the particle physics side later on in Sec. 4. However, one has to keep in mind that also certain astrophysical scenarios could lead to further restrictions or incompatibilities.

3 FN charges for the right-handed neutrinos

The first question we want to answer is how to assign FN charges to the right-handed neutrino fields. The most important constraint to keep in mind is the strong hierarchy in the right-handed (sterile) neutrino sector: In a framework as in Ref. [10], where the lightest heavy neutrino is supposed to have a mass M_1 of a few keV, the second to lightest (heavy) neutrino must at least have a mass M_2 of order GeV, due to the requirement of sufficient entropy production. This condition does, however, not constrain the largest mass M_3 , which can hence be similar to or even much larger than M_2 . In any case, we require a hierarchy of at least six orders of magnitude between M_1 and M_2 .

If we suppose that all the coefficients in the matrix M_R are of the same order, then the explanation of the HS in the sterile neutrino sector should come from the FN charges. Considering the FN parameter λ to be of the size of the Cabibbo angle [29] ($\lambda \simeq 0.22$) and $g_1 \geq g_2 \geq g_3$, the minimal conditions to be fulfilled are:

$$\begin{cases} g_1 \geq g_1|_{\min} & \text{with } g_1|_{\min} = g_2 + 3, \\ g_2 \geq g_2|_{\min} & \text{with } g_2|_{\min} = g_3. \end{cases}$$

In the minimal case, $g_2 = g_3 = g_1 - 3$, we would obtain a mass spectrum of the following type: $M_1 \simeq \mathcal{O}(\text{keV})$ and $M_2, M_3 \simeq \mathcal{O}(\text{GeV})$. For $g_2 = g_1 - 3$ and $g_3 < g_2$, the mass spectrum would instead be given by: $M_1 \simeq \mathcal{O}(\text{keV})$, $M_2 \simeq \mathcal{O}(\text{GeV})$, and $M_3 > \mathcal{O}(\text{GeV})$. Finally, if $g_2 < g_1 - 3$ and $g_3 < g_2$, both M_2 and M_3 have a mass greater than $\mathcal{O}(\text{GeV})$.

Since we want to stick to minimal choices of FN charge assignments, a reasonable condition would be to set $g_3 = 0$ and $g_1 = g_2 + 3$. Accordingly, we choose two minimal scenarios that we will investigate further:

- Scenario A: $(g_1, g_2, g_3) = (3, 0, 0)$,
- Scenario B: $(g_1, g_2, g_3) = (4, 1, 0)$.

These scenarios lead to mass eigenvalues $M_{1,2,3}$ which obey the hierarchies $M_1 \approx 10^{-6} M_{2,3}$ and $M_1 \approx 10^{-6} M_2 \approx 10^{-8} M_3$, respectively. In Tabs. 2 and 3, we show some examples of allowed textures that lead to the desired HS, considering two, three, or four independent parameters in the mass matrix M_R .

As explained before, one could increase the splitting of the mass eigenvalues by assuming stronger hierarchies in the FN charges. Furthermore, one could in principle also assign a non-zero g_3 , which would decrease the values of the masses compared to some characteristic scale that could, e.g., be generated by the VEV of some scalar field.

In Fig. 2, we have schematically depicted the general effect of the FN mechanism: A certain mass scale M_0 is multiplied by powers of λ that depend on the fermion generation. These factors lead to suppressions of the physical mass eigenvalues. In general, FN assignments are very well suited to explain strong hierarchies, which is why we ultimately chose to investigate this framework.

In the following, we will investigate only the two exemplifying Scenarios A and B, and we will show how to implement their assignments in a more complete model.

M_R	Eigenvalues
$\begin{pmatrix} A\lambda^6 & A\lambda^3 & A\lambda^3 \\ A\lambda^3 & A & B \\ A\lambda^3 & B & A \end{pmatrix}$	$M_1 = \mathcal{O}(\lambda^6) \simeq \mathcal{O}(\text{keV})$ $M_2 = A - B \simeq \mathcal{O}(\text{GeV})$ $M_3 = A + B \simeq \mathcal{O}(\text{GeV})$
$\begin{pmatrix} A\lambda^6 & A\lambda^3 & A\lambda^3 \\ A\lambda^3 & B & C \\ A\lambda^3 & C & B \end{pmatrix}$	$M_1 = \mathcal{O}(\lambda^6) \simeq \mathcal{O}(\text{keV})$ $M_2 = B - C \simeq \mathcal{O}(\text{GeV})$ $M_3 = B + C \simeq \mathcal{O}(\text{GeV})$
$\begin{pmatrix} A\lambda^6 & B\lambda^3 & B\lambda^3 \\ B\lambda^3 & C & D \\ B\lambda^3 & D & C \end{pmatrix}$	$M_1 = \mathcal{O}(\lambda^6) \simeq \mathcal{O}(\text{keV})$ $M_2 = C - D \simeq \mathcal{O}(\text{GeV})$ $M_3 = C + D \simeq \mathcal{O}(\text{GeV})$

Table 2: Examples of M_R textures for the Scenario A that lead to a HS. We have assumed all the parameters to be greater than zero.

M_R	Eigenvalues
$\begin{pmatrix} A\lambda^8 & A\lambda^5 & A\lambda^4 \\ A\lambda^5 & A\lambda^2 & B\lambda \\ A\lambda^4 & B\lambda & A \end{pmatrix}$	$M_1 = \mathcal{O}(\lambda^8) \simeq \mathcal{O}(\text{keV})$ $M_2 = \mathcal{O}(\lambda^2) \simeq \mathcal{O}(\text{GeV})$ $M_3 = A \simeq \mathcal{O}(100 \text{ GeV})$
$\begin{pmatrix} A\lambda^8 & A\lambda^5 & A\lambda^4 \\ A\lambda^5 & B\lambda^2 & C\lambda \\ A\lambda^4 & C\lambda & B \end{pmatrix}$	$M_1 = \mathcal{O}(\lambda^8) \simeq \mathcal{O}(\text{keV})$ $M_2 = \mathcal{O}(\lambda^2) \simeq \mathcal{O}(\text{GeV})$ $M_3 = B \simeq \mathcal{O}(100 \text{ GeV})$
$\begin{pmatrix} A\lambda^8 & B\lambda^5 & B\lambda^4 \\ B\lambda^5 & C\lambda^2 & D\lambda \\ B\lambda^4 & D\lambda & C \end{pmatrix}$	$M_1 = \mathcal{O}(\lambda^8) \simeq \mathcal{O}(\text{keV})$ $M_2 = \mathcal{O}(\lambda^2) \simeq \mathcal{O}(\text{GeV})$ $M_3 = C \simeq \mathcal{O}(100 \text{ GeV})$

Table 3: Examples of M_R textures for the Scenario B that lead to a HS. We have assumed all the parameters to be greater than zero.

4 Requirements, No-Go's, and amenities

In this section, we will explain in a careful way why several FN scenarios are not able to successfully produce a neutrino sector compatible with the data, or at least have problems with it. In fact, even though the FN framework seems to involve quite some freedom, the way how the charges are combined is nevertheless quite restrictive: By fixing only 9 charges (lepton doublets, right-handed electrons, and right-handed neutrinos), we predict the magnitudes of $(9 + 9 + 6) = 24$ (type I seesaw) or $(9 + 9 + 6 + 6) = 30$ (type II seesaw) entries in the matrices M_e , m_D , M_R , and m_L , which would otherwise be independent.

Accordingly, there can be many allowed theoretical and phenomenological scenarios that, when implemented within a FN framework, may lead to contradictions. In the following, we will discuss several situations which can lead to problems, as well as certain requirements that have to be fulfilled, thereby reducing step by step the arbitrariness involved in the FN models. Furthermore, we will point out why certain constraints that are typically problematic for FN inspired models do not apply in our case. As it will turn out, although

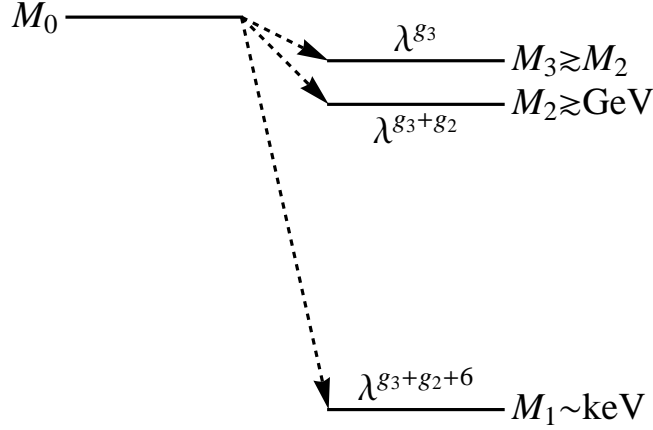


Figure 2: The mass shifting scheme of Froggatt-Nielsen models.

FN charge assignments might seem relatively random at the first sight, they are in fact restrictive enough to disagree with several frameworks, and in particular they disagree with the Left-Right symmetric framework for keV sterile neutrino Dark Matter proposed in Ref. [10], as well as with the bimaximal mixing from the neutrino side proposed in this context in Ref. [15]. Turning this round, it will be easy to rule out the FN framework if evidence for one of the contradicting scenarios is found.

Note that one could, in principle, apply additional flavour symmetries to force some entries in the mass matrices to zero. This method could alter all conclusions in this section, but such an approach is, on the other hand, far away from being minimalistic. In particular, the FN mechanism might not even be necessary anymore in such cases if, e.g., some VEV hierarchies are imposed or if the symmetry is softly broken. Since we want to stick to FN situations, however, we do not consider such extended scenarios in this paper.

4.1 No bimaximal neutrino mixing

Let us now investigate the compatibility of certain scenarios with the FN framework. One easy scenario that leads to a tri-bimaximal PMNS matrix [36] is given by a bimaximal neutrino mixing and an opportune form of the charged lepton mixing, as discussed in Ref. [37]. When trying to implement this scenario with a FN mechanism, one however encounters problems. Indeed, the assumption of bimaximal neutrino mixing poses strong constraints on the form of the light neutrino mass matrix, which can, most generally¹, look

¹Actually, there could be a second contribution that is exactly proportional to a unit matrix. Such a contribution, however, could never be explained by the FN mechanism, which can only explain orders of magnitude, but no exact equalities. Accordingly, this even more general case is not of practical relevance for FN inspired models.

like [38]:

$$M_\nu^{\text{bi-max}} = \begin{pmatrix} 0 & A & B \\ \bullet & 0 & 0 \\ \bullet & \bullet & 0 \end{pmatrix}. \quad (8)$$

Comparing this form with Eqs. (5) and (7), we can see that the FN charges (f_1, f_2, f_3) of the left-handed leptons need to be such that $|f_1 + f_{2,3}|$ is small, while all other combinations, and in particular $|2f_i|$, have to be large. This enforces large and more or less equal absolute values of all f_i 's, while simultaneously $\text{sign}(f_1) = -\text{sign}(f_{2,3})$ must be fulfilled. This is a very specific choice that cannot be brought in agreement with any of the possible forms of the charged lepton mass matrix M_e , listed in Ref. [38], which could lead to a leptonic mixing in agreement with the experimental values. We have checked that these conclusions are not significantly altered by the use of two flavon fields.

4.2 No Left-Right Symmetry

As we have just seen, it is relatively easy to impose requirements on the FN charges that are too restrictive to be fulfilled. This problem originates from the fact that it is non-trivial to obtain large mixings, as required for leptons, from a (non-lopsided) FN framework [39]. As already seen in Sec. 2, the symmetric form of the Majorana mass matrix essentially leads to some cascade-like structure at best, which can nevertheless lead to tri-bimaximal leptonic mixing in case that also the charged lepton mass matrix has a cascade-like form [27]. It is exactly this last requirement, however, that is spoiled by Left-Right (LR -) symmetry.

Imposing LR -symmetry and including Higgs triplets [40] in order to accommodate for the type II seesaw situation that is required in that context [10], the most general leptonic mass matrices must have the forms

$$\begin{aligned} (m_D)_{ij} &= v_1 f_{ij} + v_2 g_{ij}, \\ (M_e)_{ij} &= v_1 g_{ij} + v_2 f_{ij}, \\ (m_L)_{ij} &= \sqrt{2} v_L h_{ij}, \text{ and} \\ (M_R)_{ij} &= \sqrt{2} v_R h_{ij}, \end{aligned} \quad (9)$$

where $v_{1,2}$ are the Higgs doublet and $v_{L,R}$ are the Higgs triplet VEVs. However, in such a model the right-handed charged leptons $\overline{e_{iR}}$ are grouped into doublets Ψ_R^i of $SU(2)_R$ together with the right-handed neutrinos $\overline{N_{iR}}$ [and hence their FN charges must be equal, $(k_1, k_2, k_3) = (g_1, g_2, g_3)$], and the discrete LR -symmetry dictates the equality of the absolute values of the FN charges between the left- and right-handed doublets, $Q(\Psi_L^i) = Q(\Psi_R^i)$ [and hence $(f_1, f_2, f_3) = (g_1, g_2, g_3)$, if we consider only positive FN charges] [28]. From these conditions, our two Scenarios A and B already fix the complete structure of the mass matrices. Hence, the most general forms for the charged lepton and for the light neutrino mass matrices in Scenarios (A,B) are given by

$$M_e \propto \begin{pmatrix} 1 & \lambda^3 & \lambda^{3,4} \\ \lambda^3 & 1 & \lambda^{0,1} \\ \lambda^{3,4} & \lambda^{0,1} & 1 \end{pmatrix} \quad \text{and} \quad m_L \propto \begin{pmatrix} \lambda^{6,8} & \lambda^{3,5} & \lambda^{3,4} \\ \bullet & \lambda^{0,2} & \lambda^{0,1} \\ \bullet & \bullet & 1 \end{pmatrix}. \quad (10)$$

Since M_e is a Dirac-type mass matrix, and since the above form is dictated directly by the choice of (g_1, g_2, g_3) , we cannot avoid a large 11-element, which clearly spoils the demanded cascade structure.² In particular, none of the assignments that we will use later in the working examples fulfills the condition $Q(\Psi_L^i) = Q(\Psi_R^i)$.

4.3 More than one FN field

It had been shown in Ref. [41] that, in a seesaw framework, it is difficult to obtain a large mixing angle scenario for leptons with only a single $U(1)_{\text{FN}}$, unless one relies on pseudo-Dirac scenarios, which essentially involves setting some elements of the mass matrices equal to zero [42].

A very interesting comparison between one or instead two flavon fields has been performed in Ref. [43] for the quark sector in an $SU(5)$ Grand Unified Theory (GUT) inspired scenario, and it has been extended in Ref. [26] to the lepton sector. These references agree that, in order to have models which can be treated analytically, only real entries in the mass matrices should be investigated. This, however, will not allow for CP violation, since all phases could be trivially rotated away. A way to accommodate non-trivial CP phases is to introduce two flavon fields $\Theta_{1,2}$ rather than only one, and to require the VEVs of these two fields to have a relative phase. In addition, $\Theta_{1,2}$ must have opposite charges under an auxiliary Z_2 parity in order for this phase to survive. A further problem of models with only one FN field is that they normally lead to small atmospheric neutrino mixing [26], and are thus incompatible with the data. For these reasons, we are going to analyze in Sec. 5 the case of two flavon fields, instead of having only one.

4.4 Anomaly cancellation in $SU(5)$

The FN charge assignments that we are going to use are coming from an $SU(5)$ GUT scenario [44], as mentioned in Ref. [26] where, however, no detailed explanation for this form was given. We will do this by having a closer look at the conditions that have to be fulfilled in order to guarantee the cancellation of dangerous anomalies.

The charge assignments used by us can be easily understood by looking at the conditions that have to be fulfilled in order to guarantee the cancellation of anomalies. Such considerations are used, for example, in the models from Refs. [45, 46]. Here we will follow the procedure outlined in Refs. [31, 47]. In an $SU(5)$ GUT model, the right-handed electron is situated together with the quark doublets and the right-handed up-like quarks in a **10**-representation, whereas the lepton doublet is grouped together with the right-handed down-type quarks in a $\overline{\mathbf{5}}$ -representation. We use the parametrizations of Ref. [31], where the authors define the FN charges of the quark doublets, of the right-handed up quarks,

²Our statement, however, only holds with certainty in a FN framework, and in particular for Majorana neutrinos, i.e., the light neutrino mass matrix is demanded to have a symmetric structure. Note that in the framework of Ref. [27], the authors suggest LR -symmetry to lead simultaneously to cascade structures in M_e and m_ν , which can be fulfilled in a non-FN context. In the cases discussed here, however, this is not possible.

and right-handed electrons, respectively, as

$$\sum_{i=1}^3 q_i = x + u, \quad \sum_{i=1}^3 u_i = x + 2u, \quad \sum_{i=1}^3 e_i = x, \quad (11)$$

while the lepton doublets and right-handed down quarks are

$$\sum_{i=1}^3 l_i = y, \quad \sum_{i=1}^3 d_i = y + v. \quad (12)$$

Anticipating the Assignments (1,2) to be used later on in Sec. 5, cf. Tab. 4, we can determine the parameters x, y, u, v from the conditions

$$\begin{aligned} k_1 + k_2 + k_3 = 5 &= x + u = x + 2u = x, \\ f_1 + f_2 + f_3 = (1, 4) &= y = y + v, \end{aligned} \quad (13)$$

which immediately lead to $u = v = 0$ (as characteristic for assignments consistent with $SU(5)$ [31]), and hence also to $x = 5$ and $y = (1, 4)$. Since we will consider a case in which the SM-like Higgs (or other Higgses) do not carry any FN charge, we also need to fulfill $0 = -z = z + u + v$ [31], which is no problem if $z = 0$. Then, we can easily satisfy the condition for anomaly cancellation:

$$A_3 = A_2 = \frac{3}{5}A_1 = \frac{1}{2}[3x + 4u + y + v] = \frac{1}{2}[3(k_1 + k_2 + k_3) + f_1 + f_2 + f_3] = (8, 9.5), \quad (14)$$

while the A'_1 vanishes for both assignments, as demanded:

$$A'_1 = \sum_{i=1}^3 (-q_i^2 + 2u_i^2 - d_i^2 + l_i^2 - e_i^2) = 0. \quad (15)$$

Note that the FN charges (g_1, g_2, g_3) of the right-handed neutrinos do not appear in the conditions displayed in Eqs. (13) and (14). They are, indeed, total singlets not only under the SM gauge group, but also under $SU(5)$, and hence they cannot contribute to any gauge anomaly. This is the key point to be able to freely implement our right-handed neutrino Scenarios A and B.

4.5 Difficulties with $SO(10)$?

We have decided to analyze in detail an $SU(5)$ inspired model. The reason for this is not only that the right-handed FN charges (g_1, g_2, g_3) drop out of the light neutrino mass matrix, as they would in any type I or II seesaw model, but also that they are essentially unconstrained, since the right-handed neutrino \overline{N}_i is a singlet $\underline{1}_i$ for each generation i in $SU(5)$ [44]. In an $SO(10)$ GUT [44], instead, the right-handed neutrino \overline{N}_i is part of the $\underline{16}_i$ representation together with all quarks and leptons of generation i [48]. This requirement would constrain the right-handed neutrino FN charges strongly, and it is therefore not clear if it is possible to find a realistic setup that reproduces all data correctly, while at the same time keeping a strong mass splitting in the right-handed neutrino sector.

4.6 Problems with democratic Yukawa matrices

From a FN model, we generally expect a waterfall structure in the charged lepton and in the neutrino mass matrices [27]. Let us suppose that a specific FN model leads to the following matrices, which have just the form that we will also obtain for both our scenarios, cf. Sec. 5, apart from overall factors:

$$M_e^\dagger M_e \propto \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda \\ \lambda^2 & \lambda & 1 \end{pmatrix}, \quad m_\nu \propto \begin{pmatrix} \lambda^2 & \lambda & \lambda \\ \lambda & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}. \quad (16)$$

Then, these matrices will be diagonalized respectively by U_e and by $U_\nu \simeq U_\lambda U_H$:

$$U_e \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & -\lambda & 1 \end{pmatrix}, \quad U_\lambda \simeq \begin{pmatrix} 1 & \lambda & \lambda \\ -\lambda & 1 & 0 \\ -\lambda & 0 & 1 \end{pmatrix}, \quad U_H \simeq \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (17)$$

with U_λ being the matrix that diagonalizes m_ν , and U_H the matrix that corrects the neutrino eigenvalues for inverted ordering. Note that the correction U_H is required due to the presence of two large and only one small mass eigenvalues, which cannot be realized in normal hierarchy. In this case the PMNS matrix,

$$U_{\text{PMNS}} \simeq \begin{pmatrix} \lambda & \lambda & 1 \\ \lambda & 1 & -\lambda \\ 1 & -\lambda & -\lambda \end{pmatrix}, \quad (18)$$

has a form which is not consistent with the data [49], as can be easily seen by calculating the value of θ_{13} . The key points of this result are a waterfall structure for the mass matrices and the presence of the matrix U_H , and thus of an inverted ordering scenario in the neutrino sector.

This simple argument shows that in general we need to go beyond the democratic Yukawa coupling assumption in a FN scenario. There is an easy way to overcome this problem, namely by considering a slightly non-democratic Yukawa hypothesis. We will demonstrate this explicitly in Sec. 5.1.2 by considering specific $SU(5)$ inspired models, while in Sec. 5.1.1 we will analyze the democratic case and find indeed a PMNS matrix of the form of Eq. (18).

4.7 No need for RGE running

The mass matrices we obtain are actually only correct at a high energy scale, like the GUT scale, where the FN charges are imposed. Since, however, the neutrino observables are measured at a low energy scale, we have to evolve the neutrino masses and mixing parameters down to that scale by renormalization group equations (RGEs) [50, 51, 52], an effect that is often dubbed as “running”. Sometimes, this step is not applied, and it is instead argued that this should be the reason why a certain model does not fit to the data [26], although this must not necessarily hold true. But in general the running has

to be taken into account, as there is no way to avoid it. We will, however, show in the following that running effects, although present, are fully negligible in our case.

First, note that the running of the charged lepton Yukawa coupling matrix Y_e and of the light neutrino mass matrix m_ν in a type I seesaw model are given by [50],

$$\begin{aligned} 16\pi^2 \frac{dY_e}{dt} &= Y_e(D_e Y_e^\dagger Y_e + D_\nu Y_D^\dagger Y_D) + (\text{flavour diagonal}), \\ 16\pi^2 \frac{dm_\nu}{dt} &= (C_e Y_e^\dagger Y_e + C_\nu Y_D^\dagger Y_D)^T m_\nu + m_\nu (C_e Y_e^\dagger Y_e + C_\nu Y_D^\dagger Y_D) + (\text{flavour diagonal}), \end{aligned} \quad (19)$$

where $t = \ln(\mu/\mu_0)$, μ is the renormalization scale, and μ_0 is the reference scale (e.g. the GUT scale) at which the input information is imposed. The coefficients $C_{e,\nu}$ and $D_{e,\nu}$ are numbers of $\mathcal{O}(1)$. The flavour diagonal terms are not displayed, since they could only lead to an overall rescaling which is not important for mass ratios, but they will not affect the mixing. Note that we have neglected subtleties like threshold effects as they are not relevant to our argumentation.

Now, since we are considering a very low-scale seesaw framework, where the right-handed neutrino mass scale could be as low as a few GeV, the seesaw formula $m_\nu = -m_D^T M_R^{-1} m_D$ together with the definition of the Dirac neutrino mass, $m_D = y_D v$, implies that the order of the Dirac Yukawa coupling must be as small as, e.g., $y_D \sim 10^{-5}$ for $m_\nu = 1$ eV and $M_R \sim 10$ GeV. The dominant entry in Y_e , however, is $y_\tau \sim 0.01$. Accordingly one can, in Eq. (19), completely neglect Y_D , and safely assume that the largest number on the right-hand side is about $0.01^2 = 0.0001$. Dividing this number by the loop factor $16\pi^2$ decreases it to a value that is even smaller than the electron Yukawa coupling. Accordingly, any flavour non-diagonal term on the right-hand sides of Eq. (19) will be small. Hence, the only effect the running can have is an overall scaling. Furthermore, due to the smallness of Y_D , one can also neglect the running of the right-handed neutrino mass matrix [50].

We can show more explicitly that the mixing angles and phases do not undergo a considerable running. The correction to the mixing angles can be estimated to be at most about [50]

$$\Delta\theta \sim \frac{1}{16\pi^2} y_\tau^2 \xi \ln\left(\frac{\mu_0}{\mu}\right), \quad (20)$$

where ξ is an enhancement or suppression factor that can be at most as large as $\xi \sim \frac{\Delta m_A^2}{\Delta m_\odot^2} \sim 25$. Hence, from Eq. (20), one can estimate the maximal correction to a mixing angle to be about 0.001, which is perfectly negligible. Similarly, the evolution of the phases is, for a general phase ϕ , roughly given by [51]

$$\Delta\phi \sim \frac{1}{16\pi^2} y_\tau^2 \frac{1}{\zeta_{ij}} \ln\left(\frac{\mu_0}{\mu}\right) \sim \frac{10^{-5}}{\zeta_{ij}}, \quad (21)$$

where $\zeta_{ij} = \frac{m_i - m_j}{m_i + m_j}$ is a function of the light neutrino masses. Obviously, this correction to the phase is also completely negligible, unless strong degeneracies in the neutrino masses lead to a very small ζ_{ij} . Such a situation, however, is practically impossible to achieve

in FN models, since any sensible FN charge assignment will always introduce hierarchies rather than degeneracies, so that we do not have to consider the running of any phases.

Accordingly, the only effect that running could have is an overall scaling of the mass matrices. Such scalings are, however, implicitly included in the prefactors of our mass matrices and, in particular, they will cancel out in mass ratios or ratios of mass squares. Furthermore, the running will practically not affect any mixing angles or CP phases. We have verified numerically for several examples that this is indeed the case. In fact, for Yukawa couplings that are not larger than about y_τ , numerical computations show no sign of running in the mixing angles in the absence of extreme degeneracies, even if we run over several orders of magnitude in energy [52].

4.8 Potential constraints from lepton flavour violation

There is also a (seemingly unrelated) problem we want to comment on, which is, from a theoretical point of view, not necessarily connected to our models, but which might nevertheless show up in practice. This is the generic phenomenon that theories beyond the SM do not easily respect flavour, and will hence tend to lead to lepton flavour violating (LFV) reactions [53, 54].

This problem arises in our case because we rely on a GUT framework which, in turn, requires a unification of the SM gauge couplings. However, with only the SM particle content, this unification does not happen [55]. In principle, one would not necessarily have to care about this problem in a FN framework, since the FN mechanism intrinsically involves the existence of a high energy sector that is not specified further, and which is assumed to have just the right properties as to make the FN mechanism work [23]. The easy way to go would be to simply assume this high energy sector, which is present anyway, to also be responsible for gauge coupling unification.

However, in practice, one would like to have a high energy sector at hand that is specified, in order to make concrete predictions. One of the known ways to achieve gauge coupling unification is to assume the presence of supersymmetry (SUSY) [55], which is often considered when talking about GUTs. But the introduction of SUSY also leads to problems connected to this theory, the prime example being generically large LFV effects [56].

These effects have been studied in the context of GUT-inspired FN models, like ours, in Ref. [39]. Even if a flavour diagonal universal slepton mass m_0 is assumed at the GUT scale, as typical for models inspired by the minimal supergravity (mSUGRA) scenario, RGE running will lead to growing flavour-violating effects at low energies. These off-diagonal elements can be estimated to have at most the size of [39, 56]

$$(\Delta m_L^2)_{ij} \sim -\frac{6 + 2a_0^2}{16\pi^2} y_D^2 m_S^2 \ln\left(\frac{\mu_0}{\mu}\right) U_{ik} U_{jk}, \quad (22)$$

where a_0 is an $\mathcal{O}(1)$ constant, $y_D \sim 10^{-5}$ is the largest Dirac Yukawa coupling (cf. Sec. 4.7), m_S is the universal scalar mass (taken to be some typical superparticle mass), and U_{rs} are elements of the PMNS matrix. Then, the maximal (if not even overestimated) value for

the branching ratio of $\mu \rightarrow e\gamma$ can only be

$$\text{Br}(\mu \rightarrow e\gamma) \simeq \frac{\alpha^3}{G_F^2} \frac{(\Delta m_L^2)_{e\mu}^2}{m_S^8} \lesssim \frac{10^{-16}}{(m_S[\text{GeV}])^4}, \quad (23)$$

where we have used $U_{ik}U_{jk} \sim 1$ and $\ln(\mu_0/\mu) \sim 10$. However, even for scalar masses of only a few 100 GeV, the branching ratio is well below the current 90% C.L. limit of the MEGA experiment, $\text{Br}(\mu^+ \rightarrow e^+\gamma) < 1.2 \cdot 10^{-11}$ [57, 58].³ Other channels for LFV are less constrained [60], and they are not expected to yield more stringent limits.

In conclusion, LFV processes make no problems in our case, since the Dirac Yukawa coupling y_D in the models under consideration is very small, which we could also have concluded directly from the fact that we can easily ignore any running, cf. Sec. 4.7. As long as one does not include yet another potentially problematic high energy sector, our models are safe from this side.

4.9 Proton decay in SU(5) GUTs

There is one final remark that is important for GUT theories: In most extensions of the Standard Model strong constraints arise from the requirements of gauge coupling unification and of perturbativity, like it was discussed in Ref. [61]. In particular, the scale M_{GUT} at which the gauge coupling unification occurs is related to the proton life-time τ_p through dimension-six operators. In supersymmetric theories, however, the leading contributions to proton decay come from dimension-five operators resulting from the exchange of coloured higgsinos and winos. These diagrams produce an extremely fast proton decay and they have been used to constraint SUSY GUT models.

The supersymmetric $SU(5)$ in its minimal version has been tightly constrained (if not even ruled out) in Ref. [62] assuming that the gauge coupling unification is satisfied and imposing the limits provided by the Super-Kamiokande detector, which are particularly strong for the $p \rightarrow K^+\bar{\nu}$ ($\tau_p > 6.7 \times 10^{32}$ years at 90% C.L. [63]) and $p \rightarrow e^+\pi^0$ channels ($\tau_p > 8.2 \times 10^{33}$ years at 90% C.L. [64]).

It is important, however, to note that even if the minimal SUSY $SU(5)$ is highly disfavoured, this does not mean that all possible non-minimal models are excluded as well, and some of them have ways to cure the proton decay problem: There exist several works in which non-minimal SUSY $SU(5)$ models have been vastly analyzed with the purpose of solving the proton decay problem. A possible way out is achieved by a more elaborate Higgs sector such that the mass of the Higgs triplet can be pushed to very heavy values, suppressing in this way dimension five operators. This is the so-called *doublet-triplet splitting problem*, see, e.g., Ref. [65] for a review. In this framework, however, it can be difficult to suppress proton decay and at the same time not to spoil gauge coupling unification. Other ways to suppress or eliminate completely dimension-five operators have been analyzed in the context of extra-dimension models, see Refs. [66, 67, 68] for five-dimensional

³Note that the next generation experiment MEG currently provides a slightly worse value, $\text{Br}(\mu^+ \rightarrow e^+\gamma) < 2.8 \cdot 10^{-11}$ at 90% C.L. [59], which is expected to improve considerably within the next few years.

SUSY $SU(5)$ theories, and in the context of a flipped $SU(5)$ model, where the up and down Yukawa couplings are reversed with respect to the standard $SU(5)$, see Ref. [69]. Flavour symmetries have also been used to suppress dangerous proton decay operators [70]. Moreover, we want to remind the reader that the proton can naturally become almost stable if M_{GUT} does not coincide with the gauge coupling unification scale [71]. Finally, small group theoretical factors can help [72].

5 $SU(5)$ inspired models with two FN fields

We now want to give explicit working examples that yield a simultaneous explanation of the low-energy neutrino and charged lepton data, as well as a working scenario with one keV-mass sterile neutrino. To do so, we start with an extension of the model presented in Ref. [26], which was based on Ref. [43] and yields reasonable agreement with data. We will then relax three of their assumptions, in order to be able to construct fully working models: We will modify the charge assignments for the right-handed neutrinos, the ones for the left-handed lepton doublets, and we will depart from the fully democratic structure of the Yukawa matrices. Furthermore, we will present explicit analytical and numerical results in Secs. 5.1 and 5.2.

Let us start by recalling the ingredients taken from Ref. [26]: First of all, the model needs two FN flavon fields $\Theta_{1,2}$ which obtain complex VEVs. Physically, we can always choose one VEV to be real, in our case $\langle\Theta_1\rangle$. Then, the decisive quantities are the ratio λ between the VEV $\langle\Theta_1\rangle$ and the high-energy scale Λ , as well as the (complex) ratio R between the VEVs:

$$\lambda = \frac{\langle\Theta_1\rangle}{\Lambda}, \quad R = \frac{\langle\Theta_1\rangle}{\langle\Theta_2\rangle} = R_0 e^{i\alpha_0}, \quad (24)$$

where R_0 and α_0 are real numbers. Furthermore, as explained in Ref. [43], it is also necessary to introduce an auxiliary Z_2 symmetry, in order for the phase α_0 in Eq. (24) to finally be responsible for CP violation. Next, we adopt the following FN charge and Z_2 assignments inspired by Refs. [48] and [43], respectively, for the FN fields and the lepton doublets, as well as for the right-handed charged leptons and neutrinos:

$$\begin{aligned} \Theta_{1,2} : & \quad (-1, -1; +, -), \\ L_{1,2,3} : & \quad (a+1, a, a; +, +, -), \\ \overline{e_{1,2,3}} : & \quad (3, 2, 0; +, +, -), \\ \overline{N_{1,2,3}} : & \quad (g_1, g_2, g_3; +, +, -), \end{aligned} \quad (25)$$

where $a = 0, 1$ (see Tab. 4). In general, we denote the FN charges of the flavon fields by (θ_1, θ_2) , the ones of the lepton doublets by f_i , and the ones of the right-handed charged leptons by k_i , as in Sec. 2. The charges g_i of the right-handed neutrinos will be chosen according to our Scenarios A and B introduced in Sec. 3. The key point is that the FN charges of the right-handed neutrinos will drop out of the light neutrino mass matrix when constructing the light neutrino mass matrix using the seesaw mechanism of type I or II,

Field	L_1	L_2	L_3
Assignment 1 ($a = 0$)	(1, +)	(0, +)	(0, -)
Assignment 2 ($a = 1$)	(2, +)	(1, +)	(1, -)

Table 4: Family and Z_2 charges for the lepton doublets.

as explained in Refs. [26, 41, 43]. This allows us to freely choose g_i without changing the appearance of the light neutrino mass matrix. It may be, however, that the charges g_i are constrained by other consistency conditions (cf. Sec. 4.5).

Before proceeding, we also want to show how the mass matrices are constructed. The most general (seesaw type II) Lagrangian that leads to masses in the lepton sector is

$$\begin{aligned}
\mathcal{L} = & - \sum_{a,b,i,j}^{a+b=k_i+f_j} Y_e^{ij} \overline{e_{iR}} H L_{jL} \lambda_1^a \lambda_2^b + h.c. - \sum_{a,b,i,j}^{a+b=g_i+f_j} Y_D^{ij} \overline{N_{iR}} \tilde{H} L_{jL} \lambda_1^a \lambda_2^b + h.c. \\
& - \sum_{a,b,i,j}^{a+b=f_i+f_j} \frac{1}{2} \overline{(L_{iL})^C} \tilde{m}_L^{ij} L_{jL} \lambda_1^a \lambda_2^b + h.c. - \sum_{a,b,i,j}^{a+b=g_i+g_j} \frac{1}{2} \overline{N_{iR}} \tilde{M}_R^{ij} (N_{jR})^C \lambda_1^a \lambda_2^b + h.c. ,
\end{aligned} \quad (26)$$

where, by using Eq. (24),

$$\lambda_1^a \lambda_2^b \equiv \left(\frac{\Theta_1}{\Lambda} \right)^a \left(\frac{\Theta_2}{\Lambda} \right)^b = \lambda^{a+b} R^b. \quad (27)$$

The matrices Y_e , Y_D , and \tilde{M}_R are the charged lepton Yukawa matrix, the Dirac neutrino Yukawa matrix, and the uncorrected right-handed Majorana neutrino mass matrix, respectively, which were all taken in Ref. [26] to have a democratic structure. In addition to that, \tilde{m}_L is the uncorrected left-handed Majorana neutrino mass matrix, which is present in type II seesaw cases. Note that the matrix elements of \tilde{M}_R and \tilde{m}_L are all of the same order, i.e., not yet corrected by FN contributions. The sums run over all possible values of a and b that fulfill the condition of full cancellation of the FN charges in each term. Furthermore, certain terms may violate the Z_2 parity and must hence be set to zero.

Using all this, we can derive the mass matrices for four different cases: Assignment 1 or 2, each combined with Scenario A or B (for type I or type II seesaw), respectively, always separated by commas in the respective equations. For the charged leptons, we obtain

$$M_e^{(1,2)} = v \begin{pmatrix} Y_e^{11} B_{2,4} \lambda^{3,4} & Y_e^{12} B_{2,4} \lambda^{2,3} & Y_e^{13} B_{0,2} R \lambda^{2,3} \\ Y_e^{21} B_{2,4} \lambda^{2,3} & Y_e^{22} B_{0,2} \lambda^{1,2} & Y_e^{23} R \lambda^{1,2} \\ Y_e^{31} R \lambda^{1,2} & 0, Y_e^{32} R \lambda & Y_e^{33} \lambda^{0,1} \end{pmatrix}. \quad (28)$$

The right-handed neutrino mass matrices for Scenarios A and B turn out to be

$$M_R^{(A,B)} = \begin{pmatrix} \tilde{M}_R^{11} B_{6,8} \lambda^{6,8} & \tilde{M}_R^{12} B_{2,4} \lambda^{3,5} & \tilde{M}_R^{13} R B_{2,4} \lambda^{3,4} \\ \bullet & \tilde{M}_R^{22} B_{0,2} \lambda^{0,2} & 0, \tilde{M}_R^{23} R \lambda \\ \bullet & \bullet & \tilde{M}_R^{33} \end{pmatrix}, \quad (29)$$

while the left-handed ones for Assignments 1 and 2 are given by

$$m_L^{(1,2)} = \begin{pmatrix} \tilde{m}_L^{11} B_{2,4} \lambda^{2,4} & \tilde{m}_L^{12} B_{0,2} \lambda^{1,3} & \tilde{m}_L^{13} R B_{0,2} \lambda^{1,3} \\ \bullet & \tilde{m}_L^{22} B_{0,2} \lambda^{0,2} & 0, \tilde{m}_L^{23} R \lambda^2 \\ \bullet & \bullet & \tilde{m}_L^{33} B_{0,2} \lambda^{0,2} \end{pmatrix}. \quad (30)$$

Finally, the Dirac mass matrices for Assignment 1 are

$$m_D^{(1A,1B)} = v \begin{pmatrix} Y_D^{11} B_4 \lambda^{4,5} & Y_D^{12} B_{2,4} \lambda^{3,4} & Y_D^{13} R B_2 \lambda^{3,4} \\ Y_D^{21} B_{0,2} \lambda^{1,2} & Y_D^{22} \lambda^{0,1} & 0, Y_D^{23} R \lambda \\ Y_D^{31} R \lambda & 0 & Y_D^{33} \end{pmatrix}, \quad (31)$$

while the ones for Assignment 2 turn out to be

$$m_D^{(2A,2B)} = v \begin{pmatrix} Y_D^{11} B_{4,6} \lambda^{5,6} & Y_D^{12} B_4 \lambda^{4,5} & Y_D^{13} R B_{2,4} \lambda^{4,5} \\ Y_D^{21} B_2 \lambda^{2,3} & Y_D^{22} B_{0,2} \lambda^{1,2} & Y_D^{23} R \lambda^{1,2} \\ Y_D^{31} R \lambda^2 & Y_D^{32} R \lambda & Y_D^{33} \lambda \end{pmatrix}, \quad (32)$$

where $v = 174$ GeV is the electroweak VEV, and $B_{2n} = 1 + R^2 + \dots + R^{2n}$. Note that, due to $g_{2,3} = 0$ in Scenario A, the 23-entry of M_R is actually forbidden by the Z_2 parity in that case.

From M_R in Eq. (29), one can immediately calculate the mass eigenvalues for the right-handed neutrinos as functions of the right-handed mass scale M_0 :

$$\begin{aligned} \text{A}(3, 0, 0) : \quad & M_1 = M_0 \lambda^6 \sqrt{2 R_0^2 (1 + R_0^4 + 2 R_0^2 \cos(2\alpha_0))}, \\ & M_2 = M_0, \\ & M_3 = M_0 (1 + \lambda^6 [1 + R_0^2 (3 \cos(2\alpha_0) + 3 R_0^2 \cos(4\alpha_0) + R_0^4 \cos(6\alpha_0))]), \\ \text{B}(4, 1, 0) : \quad & M_1 = M_0 \lambda^8 \sqrt{2 R_0^4 (1 + R_0^8 - 2 R_0^4 \cos(4\alpha_0))}, \\ & M_2 = M_0 \lambda^2, \\ & M_3 = M_0 (1 + R_0^2 \lambda^2 \cos(2\alpha_0)). \end{aligned}$$

Indeed, the eigenvalues of M_R show just the hierarchical structure that we have expected: If we suppose that M_1 is of $\mathcal{O}(\text{keV})$, then M_0 should be about $10^6 \text{ keV} \sim 1 \text{ GeV}$ for Scenario A, or about $10^8 \text{ keV} \sim 100 \text{ GeV}$ for Scenario B. In any case, we would have a low-scale seesaw to work. One can of course raise the possible value for M_0 by simply assigning an even higher charge g_1 to the first generation right-handed neutrino: Already for $g_1 = 5$, one could have $M_0 \sim 10 \text{ TeV}$, while one would only have to be careful to keep $g_1 - g_2 \geq 3$, as explained in Sec. 3.

5.1 Analytical results

In this section, we will exemplify for Assignment 1 (and seesaw type I), how one can arrive at analytical approximations for the masses and for the PMNS-matrix. We will also

see explicitly that it is necessary to depart from the democratic structure of the Yukawa matrices in order to obtain sensible results (cf. Sec. 4.6), i.e., the FN mechanism alone is not strong enough to fully explain the data. It will turn out, however, that this simple analytical consideration is still not perfect, even in the non-democratic case, and we have to rely on numerics in order to find quasi-perfect models, which will be discussed in Sec. 5.2.

5.1.1 Democratic matrices

The natural starting point is having democratic forms of all Yukawa and bare mass matrices, i.e.,

$$Y_e^{ij} = y_e, \quad Y_D^{ij} = y_D, \quad \text{and} \quad \tilde{M}_R^{ij} = M_0 \quad \forall i, j. \quad (33)$$

In this case, the type I light neutrino mass matrices can be easily calculated from Eqs. (29) and (31), using the seesaw formula $m_\nu = -m_D^T M_R^{-1} m_D$. For Scenario A, this results in

$$m_\nu^{(1A),I} = m_{\nu 0} \begin{pmatrix} \frac{\lambda^2(-2B_2B_4(R^2+1)+B_4^2+B_6R^2+B_6)}{B_2^2(R^2+1)-B_6} & -\lambda & -R\lambda \\ -\lambda & -1 & 0 \\ -R\lambda & 0 & -1 \end{pmatrix}, \quad (34)$$

whereas the corresponding expression for Scenario B is given by

$$m_\nu^{(1B),I} = m_{\nu 0} \begin{pmatrix} -B_2\lambda^2 & -\lambda & -R\lambda \\ -\lambda & \frac{(-B_2^2R^2+B_4^2(B_2-R^2-2)+2B_2B_4R^2+B_8)}{B_2^3R^2-2B_2B_4R^2-B_2B_8+B_4^2+B_8R^2} & 0 \\ -R\lambda & 0 & -1 \end{pmatrix}, \quad (35)$$

where $m_{\nu 0} = \frac{y_D^2 v^2}{M_0}$ in both cases. In order to obtain the corresponding mixing, we need to diagonalize the two neutrino mass matrices in Eqs. (34) and (35), as well as the charged lepton mass matrix for Assignment 1 from Eq. (28). It turns out that all these matrices can be approximately diagonalized by applying a series of small and large rotation matrices, all of which are (at least approximately) unitary. Such a stepwise diagonalization, although certainly not suitable for general mass matrices, is ideally suited for FN models, since the corresponding matrices intrinsically involve the small parameter λ in which all rotation matrices can be Taylor expanded. Note that, in order to derive the leptonic mixing matrices, we follow the conventions used in the *Mixing Parameter Tools (MPT)* package [50, 73], which will also be used later on for our numerical computations.

In order to obtain the (unitary) charged lepton mixing matrix U_e , we have to diagonalize the squared charged lepton mass matrix $M_e^\dagger M_e$ by

$$U_e^\dagger M_e^\dagger M_e U_e = \text{diag}(m_e^2, m_\mu^2, m_\tau^2). \quad (36)$$

In our case, Eq. (28), it turns out that the charged lepton mixing matrix is given by

$$U_e = U_A U_B U_C U_D U_E U_F U_G, \quad (37)$$

where the respective matrices are all at least approximately unitary and are reported in Eq. (A-1). These subsequent transformations bring $M_e^\dagger M_e$ to an approximately diagonal form, from which one can read off the expressions for the charged lepton masses to be

$$\begin{cases} m_e &= m_0 \lambda^3 R_0^2, \\ m_\mu &= m_0 \lambda \left(1 + \lambda^2 \left[R_0^2 \cos(2\alpha_0) + \frac{R_0^4 - R_0^2 + 3}{2} \right] \right), \\ m_\tau &= m_0 \left(1 + \frac{3}{2} R_0^2 \lambda^2 \right). \end{cases}$$

From these relations, we can determine the mass ratios: $m_e/m_\mu \simeq R_0^2 \lambda^2$ and $m_\mu/m_\tau \simeq \lambda$. Using the measured values of m_e , m_μ , and m_τ , we find the sizes of the parameters to be roughly

$$\lambda \simeq 0.06 \quad \text{and} \quad R_0 \simeq 1.18. \quad (38)$$

As noted in Ref. [39], the most advantageous choice of the parameter λ turns out to be a bit below the standard choice 0.22. The phase α_0 is not fixed by the lowest order expressions in Eq. (38), but one can choose it to have the value $\alpha_0 = 0.67$ in order to make the $\mathcal{O}(\lambda^3)$ -correction to m_μ/m_τ vanish.⁴

Let us now have a look at the neutrino mass matrix for Scenario A. The matrix $m_\nu^{(1A),I}$ in Eq. (34) can be diagonalized by a unitary matrix $U_\nu \equiv U_\nu^{(1A),I}$ according to $U_\nu^T m_\nu^{(1A),I} U_\nu = \text{diag}(m_1, m_2, m_3)$, where the mass eigenvalues m_i can still contain complex phases. Also this can be done by a stepwise diagonalization resulting in

$$U_\nu = U_\alpha U_\beta U_\gamma U_\delta U_\epsilon U_\zeta U_\eta, \quad (39)$$

with the respective pieces being all at least approximately unitary and are given by Eq. (A-3). Note that the purpose of the last matrix U_η is only to reshuffle the mass eigenvalues in order to accommodate for the correct ordering, since the resulting pattern (two larger eigenvalues and one smaller one) can only be realized in inverted ordering. It is exactly this point, which will change in the non-democratic cases, and this is also the reason why the democratic cases provide a worse match to the data. The neutrino mass eigenvalues can be determined as the absolute values of the diagonal entries of the resulting mass matrix. They turn out to be

$$\begin{cases} |m_1| &= m_{\nu 0} [1 + \lambda^2(1 + R_0^2 - \sqrt{(-1 + R_0^2)^2 + R_0^2 \cos^2 \alpha_0})] + \mathcal{O}(\lambda^4), \\ |m_2| &= m_{\nu 0} [1 + \lambda^2(1 + R_0^2 + \sqrt{(-1 + R_0^2)^2 + R_0^2 \cos^2 \alpha_0})] + \mathcal{O}(\lambda^4), \\ |m_3| &= m_{\nu 0} \lambda^2 \frac{R_0^2}{2\sqrt{R_0^4 + 2R_0^2 \cos(2\alpha_0) + 1}} + \mathcal{O}(\lambda^4). \end{cases}$$

The mass square differences are, at lowest order,

$$\begin{cases} \Delta m_\odot^2 &\simeq 4m_{\nu 0}^2 \lambda^2 \sqrt{(-1 + R_0^2)^2 + R_0^2 \cos^2 \alpha_0} + \mathcal{O}(\lambda^4), \\ \Delta m_A^2 &\simeq m_{\nu 0}^2 (1 + 2\lambda^2[1 + R_0^2 - \sqrt{(-1 + R_0^2)^2 + R_0^2 \cos^2 \alpha_0}]) + \mathcal{O}(\lambda^4). \end{cases}$$

⁴For Assignment 2, we would have obtained $m_e/m_\mu \simeq R_0^4 \sqrt{1 + R_0^2} \lambda^2$ and $m_\mu/m_\tau \simeq \frac{\lambda}{1 + R_0^2}$, leading to $\lambda \simeq 0.10$ and $R_0 \simeq 0.80$. The phase α_0 is unconstrained, but the choice $\alpha_0 = 0.60$ turns out to be numerically convenient.

Using the numbers from Eq. (38), one can predict the ratio $\frac{\Delta m_\odot^2}{\Delta m_A^2} \simeq [0.018, 0.005]$, respectively for $\alpha_0 = [0, \pi/2]$, where the choice $\alpha_0 = 0.67$ leads to the value 0.015. To obtain a fair agreement with the measured value of about 0.031 we would need $\alpha_0 = 0$ and thus no CP violation. In addition to the imperfect prediction for $\frac{\Delta m_\odot^2}{\Delta m_A^2}$, the democratic form of the Yukawa matrices does lead to a bad mixing. This fact is visible when looking at the full PMNS-matrix, $U_{\text{PMNS}} = U_e^\dagger U_\nu$, which is given by

$$U_{\text{PMNS}}^{(1A),I} = \begin{pmatrix} 0 & 0 & 1 + (R_0^2 + 1)\lambda^2 \\ \frac{-\zeta_1}{\sqrt{R_0^2 + \zeta_1^2}} + u_{21}^{(1A),I}\lambda^2 & \frac{\zeta_1}{\sqrt{R_0^2 + \zeta_1^2}} + u_{21}^{(1A),I}\lambda^2 & 0 \\ \frac{R_0}{\sqrt{R_0^2 + \zeta_1^2}} + u_{31}^{(1A),I}\lambda^2 & \frac{R_0}{\sqrt{R_0^2 + \zeta_1^2}} + u_{32}^{(1A),I}\lambda^2 & 0 \end{pmatrix} + \mathcal{O}(\lambda^3), \quad (40)$$

where

$$u_{21}^{(1A),I} = \frac{-R_0^2(2\cos\alpha_0 + i\sin\alpha_0 - 2\zeta_1)}{2\sqrt{R_0^2 + \zeta_1^2}}, \quad u_{31}^{(1A),I} = \frac{R_0(2R_0^2 + \zeta_1[i\sin\alpha_0 - 2\cos\alpha_0])}{2\sqrt{R_0^2 + \zeta_1^2}},$$

$$u_{32}^{(1A),I} = \frac{R_0(2R_0^2 + \zeta_1[2\cos\alpha_0 - i\sin\alpha_0])}{2\sqrt{R_0^2 + \zeta_1^2}}, \quad (41)$$

and ζ_1 is defined in Eq. (A-4). Obviously, this matrix does not have the desired form, since the angle θ_{13} is extremely close to the maximal value $\pi/2$, instead of being small or even vanishing.

Let us now check if Scenario B can give a better match with data. Here, the unitary matrix $U_\nu \equiv U_\nu^{(1B),I}$ is given by

$$U_\nu = U'_\alpha U'_\beta U'_\gamma U'_\delta U'_\epsilon U'_\zeta, \quad (42)$$

where the individual rotation matrices are reported in Eq. (A-5). Again, the last matrix U'_ζ corrects for inverted ordering. The eigenvalues read

$$\begin{cases} |m_1| = m_{\nu 0} (1 + 2R_0^2\lambda^2) + \mathcal{O}(\lambda^4), \\ |m_2| = m_{\nu 0} \frac{(R_0^4(\lambda^2+2) - 2R_0^2\lambda^2\cos(2\alpha_0) + \lambda^2)}{R_0^2\sqrt{R_0^4 - 2R_0^2\cos(2\alpha_0) + 1}} + \mathcal{O}(\lambda^4), \\ |m_3| = m_{\nu 0}\lambda^2 \frac{\sqrt{R_0^4 + 2R_0^2\cos(2\alpha_0) + 1}}{2R_0^2} + \mathcal{O}(\lambda^4), \end{cases}$$

and the mass square differences are, at lowest order,

$$\begin{cases} \Delta m_\odot^2 \simeq m_{\nu 0}^2 \left(-4(R_0^2 - 1)\lambda^2 + \frac{4R_0^4}{R_0^4 - 2R_0^2\cos(2\alpha_0) + 1} - 1 \right) + \mathcal{O}(\lambda^4), \\ \Delta m_A^2 \simeq m_{\nu 0}^2 (1 + 4R_0^2\lambda^2) + \mathcal{O}(\lambda^4). \end{cases}$$

Then, even with the most suitable value of $\alpha_0 = \frac{\pi}{2}$, the ratio of mass square differences,

$\frac{\Delta m_{\odot}^2}{\Delta m_A^2} \simeq 0.35$, turns out to be much too large. Also the PMNS matrix,

$$U_{\text{PMNS}}^{(1B),I} = \begin{pmatrix} 0 & -\frac{i(R_0^2 + e^{2i\alpha_0})}{2R_0^2}\lambda & 1 + \left(R_0^2 + \frac{1}{2} - \frac{e^{-2i\alpha_0}}{2R_0^2}\right)\lambda^2 \\ u_{21}^{(1B),I}\lambda^2 & \frac{1}{2}i\left(2 + \left(1 - \frac{e^{2i\alpha_0}}{R_0^2}\right)\lambda^2\right) & \frac{1}{2}\left(1 + \frac{e^{-2i\alpha_0}}{R_0^2}\right)\lambda \\ i(1 + R_0^2\lambda^2) & \frac{2ie^{-i\alpha_0}R_0^3(-e^{4i\alpha_0} + 2R_0^2 + e^{2i\alpha_0}R_0^2)}{3R_0^4 + 2\cos(2\alpha_0)R_0^2 - 1}\lambda^2 & 0 \end{pmatrix} + \mathcal{O}(\lambda^3), \quad (43)$$

where

$$u_{21}^{(1B),I} = -\frac{2ie^{-3i\alpha_0}R_0^3[-1 + e^{2i\alpha_0}(1 + 2e^{2i\alpha_0}R_0^2)]}{3R_0^4 + 2\cos(2\alpha_0)R_0^2 - 1}, \quad (44)$$

turns out not to be much better than the one from Eq. (40). From the above calculations it is clear that the hypothesis of democratic Yukawa matrices does not lead to a satisfactory PMNS matrix. In the following section, we will analyze the non-democratic case to see if we might be able to achieve better agreement with the neutrino data.

5.1.2 Slightly non-democratic matrices

The problem of fully democratic matrices not being perfectly suitable for $SU(5)$ inspired FN models has already been discussed in Ref. [74]. The key point to obtain a better agreement with data is to change the form of the light neutrino mass matrix. Ideally, they should have the form

$$m_\nu \propto \begin{pmatrix} \lambda^2 & \lambda & \lambda \\ \lambda & \delta_0^2 & 0 \\ \lambda & 0 & 1 \end{pmatrix}, \quad (45)$$

with $\delta_0^2 = \sqrt{\frac{\Delta m_{\odot}^2}{\Delta m_A^2}} \simeq 0.18 > \lambda$, whereas our matrices in Eqs. (34) and (35) instead have the form

$$m_\nu \propto \begin{pmatrix} \lambda^2 & \lambda & \lambda \\ \lambda & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}, \quad (46)$$

which looks similar, but causes a major difference for the mass eigenvalues and for the mixings. A suitable modification can be obtained by simply choosing the Yukawa couplings $Y_D^{12} = Y_D^{22} = \delta_0 y_D$ instead of $Y_D^{12} = Y_D^{22} = y_D$. Indeed, due to $\delta_0 \sim 0.30$, this is just a relatively mild deviation from the fully democratic case, and it can easily be justified by varying the entries in the Yukawa matrices in a certain range around their natural values y_D .

Using this modification, the new light neutrino masses look like

$$m_\nu^{(1A),I}|_{\text{non-d.}} = m_{\nu 0} \begin{pmatrix} \frac{\lambda^2(-2B_2B_4(R^2+1)+B_4^2+B_6R^2+B_6)}{B_2^2(R^2+1)-B_6} & -\delta_0\lambda & -R\lambda \\ -\delta_0\lambda & -\delta_0^2 & 0 \\ -R\lambda & 0 & -1 \end{pmatrix}, \quad (47)$$

$$m_\nu^{(1B),I}|_{\text{non-d.}} = m_{\nu 0} \begin{pmatrix} -B_2 \lambda^2 & -\delta_0 \lambda & -R \lambda \\ -\delta_0 \lambda & \frac{\delta_0^2 (-B_2^2 R^2 + B_4^2 (B_2 - R^2 - 2) + 2B_2 B_4 R^2 + B_8)}{B_2^3 R^2 - 2B_2 B_4 R^2 - B_2 B_8 + B_4^2 + B_8 R^2} & 0 \\ -R \lambda & 0 & -1 \end{pmatrix}. \quad (48)$$

The matrix in Eq. (47) can again be diagonalized by a unitary matrix U_ν , which is now given by

$$U_\nu = U_\alpha U_\beta U_\gamma U_\delta U_\epsilon, \quad (49)$$

with the respective pieces reported in Eq. (A-7). Note that now, there is no need to correct the mass ordering, since the δ_0^2 -term lowers the second eigenvalue down such that the mass square difference between the first and the second squared eigenvalues can now be as small as the measured value of Δm_\odot^2 . The neutrino mass eigenvalues can again be determined as the absolute values of the diagonal entries of the resulting mass matrix. They turn out to be

$$\begin{cases} |m_1| &= m_{\nu 0} \lambda^2 \frac{R_0^2}{2\sqrt{R_0^4 + 2R_0^2 \cos(2\alpha_0) + 1}} + \mathcal{O}(\lambda^4), \\ |m_2| &= m_{\nu 0} (\delta_0^2 + 2\lambda^2) + \mathcal{O}(\lambda^4), \\ |m_3| &= m_{\nu 0} (1 + 2R_0^2 \lambda^2) + \mathcal{O}(\lambda^4). \end{cases}$$

The mass square differences are, at lowest order,

$$\begin{cases} \Delta m_\odot^2 &\simeq m_{\nu 0}^2 \delta_0^2 (\delta_0^2 + 4\lambda^2) + \mathcal{O}(\lambda^4), \\ \Delta m_A^2 &\simeq m_{\nu 0}^2 (1 + 4R_0^2 \lambda^2) + \mathcal{O}(\lambda^4). \end{cases}$$

Using the numbers from Eq. (38), as well as $\delta_0^2 \simeq 0.18$, we now predict the ratio $\frac{\Delta m_\odot^2}{\Delta m_A^2} \simeq 0.034$, which is a quasi perfect match to the data. In fact, the trick is that the ratio between the mass square differences is just given by δ_0^4 , which precisely justifies the choice made for δ_0^2 . The PMNS-matrix turns out to be

$$U_{\text{PMNS}}^{(1A),I} = \begin{pmatrix} 1 + (R_0^2 + \frac{1}{\delta}) \lambda^2 & -\frac{i(\delta_0 - 1)\lambda}{\delta_0} & 0 \\ \frac{(\delta_0 - 1)\lambda}{\delta_0} & i(1 + \frac{\lambda^2}{\delta_0}) & u_{23}^{(1A),I} \lambda^2 \\ 0 & \frac{iR_0 e^{-i\alpha_0} (-1 + \delta_0 + e^{2i\alpha_0} \delta_0^3 + \delta_0^4)}{\delta_0^4 - 1} \lambda^2 & i(1 + R_0^2 \lambda^2) \end{pmatrix} + \mathcal{O}(\lambda^3), \quad (50)$$

where $u_{23}^{(1A),I} = R_0 \left(-\frac{i(\delta_0^2 + \delta_0 - 1) \cos \alpha_0}{\delta_0^2 - 1} + \sin \alpha_0 - \frac{\delta_0 \sin \alpha_0}{\delta_0^2 + 1} \right)$. Although this matrix is still not really perfect, the value of the angle θ_{13} is much better than before, since $\theta_{13} \simeq \mathcal{O}(\lambda^3)$. Hence, we can be optimistic to find quasi-perfect numerical models when perturbing the non-democratic mass matrices.

Finally, for the non-democratic Scenario B, we have a neutrino mixing matrix given by

$$U_\nu = U'_\alpha U'_\beta U'_\gamma U'_\delta U'_\epsilon, \quad (51)$$

with the respective pieces given in Eq. (A-9). Also in this case, there is no need to correct for inverted ordering. The eigenvalues read

$$\begin{cases} |m_1| &= m_{\nu 0} \lambda^2 \frac{\sqrt{R_0^4 + 2R_0^2 \cos(2\alpha_0) + 1}}{2R_0^2} + \mathcal{O}(\lambda^4), \\ |m_2| &= m_{\nu 0} \frac{(R_0^4(2\delta_0^2 + \lambda^2) - 2R_0^2 \lambda^2 \cos(2\alpha_0) + \lambda^2)}{R_0^2 \sqrt{R_0^4 - 2R_0^2 \cos(2\alpha_0) + 1}} + \mathcal{O}(\lambda^4), \\ |m_3| &= m_{\nu 0} (1 + 2R_0^2 \lambda^2) + \mathcal{O}(\lambda^4), \end{cases}$$

and the mass square differences are, at lowest order,

$$\begin{cases} \Delta m_{\odot}^2 &\simeq 4m_{\nu 0}^2 \delta_0^2 \left(\lambda^2 + \frac{\delta_0^2 R_0^4}{R_0^4 - 2R_0^2 \cos(2\alpha_0) + 1} \right) + \mathcal{O}(\lambda^4), \\ \Delta m_A^2 &\simeq m_{\nu 0}^2 (1 + 4R_0^2 \lambda^2) + \mathcal{O}(\lambda^4). \end{cases}$$

Again taking the most suitable value of $\alpha_0 = \frac{\pi}{2}$, the ratio of mass square differences, $\frac{\Delta m_{\odot}^2}{\Delta m_A^2}$, is now 0.046, which is still too large but much closer to the actual value than before. The PMNS matrix is given by

$$U_{\text{PMNS}}^{(1B),I} = \begin{pmatrix} 1 + \left(R_0^2 + \frac{1}{2\delta_0} - \frac{e^{-2i\alpha_0}}{2\delta_0 R_0^2} \right) \lambda^2 & -\frac{i((2\delta_0 - 1)R_0^2 + e^{2i\alpha_0})}{2R_0^2 \delta_0} \lambda & 0 \\ \left(1 + \frac{e^{-2i\alpha_0} - R_0^2}{2R_0^2 \delta_0} \right) \lambda & i \left(1 + \frac{(R_0^2 - e^{2i\alpha_0})}{2R_0^2 \delta_0} \right) \lambda^2 & u_{23}^{(1B),I} \lambda^2 \\ 0 & u_{32}^{(1B),I} \lambda^2 & i(1 + R_0^2 \lambda^2) \end{pmatrix} + \mathcal{O}(\lambda^3), \quad (52)$$

where

$$\begin{aligned} u_{23}^{(1B),I} &= R_0 \left(-ie^{-i\alpha_0} + \frac{n_{23}}{(4\delta_0^4 - 1)R_0^4 + 2\cos(2\alpha_0)R_0^2 - 1} \right), \\ n_{23} &= \delta_0 (2e^{-2i\alpha_0} \delta_0^2 R_0^2 + R_0^2 - e^{2i\alpha_0}) ((R_0^2 + 1) \sin \alpha_0 - i(R_0^2 - 1) \cos \alpha_0), \\ u_{32}^{(1B),I} &= \frac{ie^{-3i\alpha_0} R_0 n_{32}}{(4\delta_0^4 - 1)R_0^4 + 2\cos(2\alpha_0)R_0^2 - 1}, \\ n_{32} &= -2e^{6i\alpha_0} R_0^2 \delta_0^3 - R_0^2 (\delta_0 - 1) + e^{4i\alpha_0} R_0^2 (2R_0^2 \delta_0^3 - \delta_0 + 1) + \\ &\quad + e^{2i\alpha_0} ((4\delta_0^4 + \delta_0 - 1)R_0^4 + \delta_0 - 1). \end{aligned} \quad (53)$$

Also in this case, the value of the angle θ_{13} is much better than in the democratic case, since $\theta_{13} \simeq \mathcal{O}(\lambda^3)$. However, to find quasi-perfect models, we will have to turn to a numerical study, which is presented in the next section.

5.2 Numerical analysis

Finally, we want to present a few quasi-perfect models and their predictions. Usually, and in particular in the context of FN inspired models, so-called scatter plots are presented in the literature. These plots are supposed to indicate that the model is consistent with data in a considerable region of the parameter space. However, we do not consider this approach

1AI	Matrix
M_e	$M_{e0} \begin{pmatrix} 0.81B_2\lambda^3 & 1.44B_2\lambda^2 & 0.29R\lambda^2 \\ 2.00B_2\lambda^2 & 1.13\lambda & 2.50R\lambda \\ 3.71R\lambda & 0 & 0.35 \end{pmatrix}$
m_D	$m_{D0} \begin{pmatrix} 0.75B_4\lambda^4 & 0.15B_2\lambda^3 & 1.42B_2R\lambda^3 \\ 0.51\lambda & 0.13 & 0 \\ 3.32R\lambda & 0 & 2.93 \end{pmatrix}$
M_R	$M_0 \begin{pmatrix} 0.38B_6\lambda^6 & 0.31B_2\lambda^3 & 1.26B_2R\lambda^3 \\ 0.31B_2\lambda^3 & 4.18 & 0 \\ 1.26B_2R\lambda^3 & 0 & 4.81 \end{pmatrix}$

Table 5: Mass matrices of model 1AI. The prefactors parametrize the mass scales imposed on the concrete model at a certain energy scale.

1BI	Matrix
M_e	$M_{e0} \begin{pmatrix} 0.91B_2\lambda^3 & 2.26B_2\lambda^2 & 4.33R\lambda^2 \\ 3.80B_2\lambda^2 & 2.51\lambda & 3.63R\lambda \\ 0.79R\lambda & 0 & 0.15 \end{pmatrix}$
m_D	$m_{D0} \begin{pmatrix} 3.20B_4\lambda^5 & 0.15B_4\lambda^4 & 3.73B_2R\lambda^4 \\ 2.27B_2\lambda^2 & 0.030\lambda & 1.23R\lambda \\ 1.69R\lambda & 0 & 0.66 \end{pmatrix}$
M_R	$M_0 \begin{pmatrix} 0.33B_8\lambda^8 & 2.68B_4\lambda^5 & 1.63B_2R\lambda^4 \\ 2.68B_4\lambda^5 & 2.10B_2\lambda^2 & 0.83R\lambda \\ 1.63B_2R\lambda^4 & 0.83R\lambda & 0.49 \end{pmatrix}$

Table 6: Mass matrices of model 1BI. The prefactors parametrize the mass scales imposed on the concrete model at a certain energy scale.

as too useful, since first such results always depend on the statistical measure used in the generation of the random numbers involved, and second because there is essentially no method to determine if indeed a large part of the parameter space is investigated or rather only a tiny patch, due to the gigantic complexity of higher-dimensional non-Cartesian spaces. Furthermore, there is no way to decide whether a model can be considered as “good” or “bad”, just because a certain fraction of random parameter choices leads to compatibility with data.

We will rather take on a contrary approach and try to find four fully working examples by perturbing the mass matrices from Eqs. (28), (29), (31), and (32), where the Dirac Yukawa coupling matrices Y_D are taken to be non-democratic, according to Sec. 5.1.2. If these examples are in agreement with data, we consider the FN approach as being predictive in the sense that small departures from the analytical forms of the mass matrices are perfectly enough to be in full agreement with low-energy neutrino data.

Based on the analytical results from Sec. 5.1.1, we choose the parameters λ , R_0 , and α_0 in the following way:

2AI	Matrix
M_e	$M_{e0} \begin{pmatrix} 4.96B_4\lambda^4 & 3.70B_2\lambda^3 & 3.63B_2R\lambda^3 \\ 4.75B_2\lambda^3 & 1.51B_2\lambda^2 & 3.42R\lambda^2 \\ 3.17R\lambda^2 & 0.70R\lambda & 0.28\lambda \end{pmatrix}$
m_D	$m_{D0} \begin{pmatrix} 3.79B_4\lambda^5 & 0.0057B_4\lambda^4 & 4.60B_2R\lambda^4 \\ 0.68B_2\lambda^2 & 0.097\lambda & 1.20R\lambda \\ 1.12R\lambda^2 & 3.30R\lambda & 4.85\lambda \end{pmatrix}$
M_R	$M_0 \begin{pmatrix} 3.02B_6\lambda^6 & 1.28B_2\lambda^3 & 0.93B_2R\lambda^3 \\ 1.28B_2\lambda^3 & 4.32 & 0 \\ 0.93B_2R\lambda^3 & 0 & 3.08 \end{pmatrix}$

Table 7: Mass matrices of model 2AI. The prefactors parametrize the mass scales imposed on the concrete model at a certain energy scale.

- Assignment 1: $\lambda = 0.06$, $R_0 = 1.18$, $\alpha_0 = 0.67$,
- Assignment 2: $\lambda = 0.10$, $R_0 = 0.80$, $\alpha_0 = 0.60$.

The next step is to generate a number of perturbed charged lepton matrices according to Eq. (28), and check which of them yield a quasi perfect prediction ($\pm 1\%$) for the charged lepton mass ratios m_e/m_μ and m_μ/m_τ . We furthermore generate right-handed neutrino mass matrices according to Eq. (29), whose smallest eigenvalue should always be at least six orders of magnitude smaller than the one of the second to lightest mass eigenstate. We then use these matrices together with the Dirac mass matrices from Eqs. (31) and (32) to generate light neutrino mass matrices according to the seesaw formula, $m_\nu = -m_D^T M_R^{-1} m_D$, which yield a quasi-perfect prediction ($\pm 1\%$) for the ratio of neutrino mass square differences, $\Delta m_\odot^2/\Delta m_A^2$. Finally we combine the lists of matrices and calculate the mixing parameters and phases using MPT [50, 73]. All models that lead to predictions of the mixing angles that are consistent with the current 3σ ranges [49] will survive the test.

The corresponding mass matrices of the quasi-perfect numerical models for all four combinations of Scenarios A and B, as well as of Assignments 1 and 2 can be found in Tabs. 5 to 8. Indeed, all mass matrices have just the structure that we have desired: Only slight perturbations of the democratic form, with the exception of the Dirac mass matrices that have smaller 12 and 22 entries. Accordingly, as we had expected, the FN models presented here are predictive up to the exact values of the mixing angles and CP phases. In particular, all charged lepton and light neutrino mass matrices lead to the correct ratios of masses or mass square differences, respectively.

The numerical predictions are summarized in Tab. 9, and the mixing angles and masses are also displayed in Fig. 3. As can be seen, all mixing angle predictions are in agreement with the 3σ ranges of the oscillation parameters [49], because the models have been selected in that way. Furthermore, as explained in Sec. 4.7, these predictions are effectively scale invariant. However, a non-trivial prediction is the actual values of the masses (in terms of the absolute neutrino mass scale m_ν , which is determined by the other scales involved), and

2BI	Matrix				
M_e	M_{e0}	$\begin{pmatrix} 2.71B_4\lambda^4 & 4.17B_2\lambda^3 & 3.74B_2R\lambda^3 \\ 1.39B_2\lambda^3 & 0.19B_2\lambda^2 & 3.19R\lambda^2 \\ 0.91R\lambda^2 & 0.38R\lambda & 0.40\lambda \end{pmatrix}$			
m_D	m_{D0}	$\begin{pmatrix} 1.57B_6\lambda^6 & 0.078B_4\lambda^5 & 1.79B_4R\lambda^5 \\ 4.87B_2\lambda^3 & 0.16B_2\lambda^2 & 4.85R\lambda^2 \\ 3.47R\lambda^2 & 1.82R\lambda & 1.25\lambda \end{pmatrix}$			
M_R	M_0	$\begin{pmatrix} 0.61B_8\lambda^8 & 1.12B_4\lambda^5 & 0.53B_2R\lambda^4 \\ 1.12B_4\lambda^5 & 3.85B_2\lambda^2 & 4.33R\lambda \\ 0.53B_2R\lambda^4 & 4.33R\lambda & 3.21 \end{pmatrix}$			

Table 8: Mass matrices of model 2BI. The prefactors parametrize the mass scales imposed on the concrete model at a certain energy scale.

Mod.	s_{12}^2	s_{13}^2	s_{23}^2	δ	α	β	m_1/m_ν	m_2/m_ν	m_3/m_ν	Hier.
1AI	0.28	0.018	0.54	4.21	0.58	1.25	0.0014	0.19	1.03	NH
1BI	0.31	0.035	0.41	4.98	2.35	2.89	0.099	0.59	0.0028	IH
2AI	0.31	0.0015	0.40	3.88	2.27	0.54	$1.7 \cdot 10^{-8}$	0.020	0.11	NH
2BI	0.27	0.021	0.57	3.83	1.96	2.71	0.0018	0.0054	0.029	NH

Table 9: Predictions of the four numerical models. “Mod.” stands for “model”, $s_{ij}^2 \equiv \sin^2 \theta_{ij}$, and “Hier.” stands for “hierarchy”.

in particular the dominant prediction of normal mass ordering (or, due to the hierarchical structure of FN matrices, normal hierarchy), which we had already anticipated in Sec. 5.1.2.

6 Conclusions

In this paper, we have shown how to carefully derive models explaining the appearance of one right-handed (sterile) neutrino that has a mass at the keV scale, while at the same time predicting leptonic mass ranges and mixing parameters in full agreement with experiments. These models were based on the famous Froggatt-Nielsen mechanism, which is a well-known possibility to create strong hierarchies between fermion masses. To our knowledge, this is the third known type of models that can successfully explain the existence of a keV sterile neutrino Dark Matter particle, the first two being based on soft breaking of flavour symmetries [14, 15] and on Randall-Sundrum warping [17], respectively, where the former exploits symmetry breaking effects to lift a massless state to the keV scale while the latter uses the exponential suppression of UV-brane physics to strongly suppress the natural right-handed neutrino mass scale. We have instead made use of the possibility to assign FN charges to the different fermions in such a way as to suppress certain mass matrix elements, whose sizes are reduced by roughly one order of magnitude per unit FN charge. In this way, it is possible to suppress one right-handed neutrino mass strongly enough to

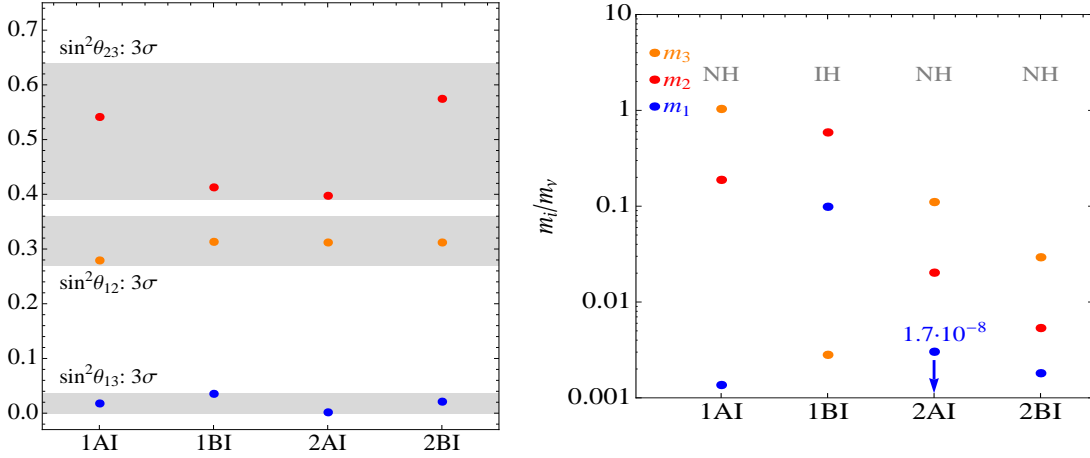


Figure 3: The predictions of the numerical models for the leptonic mixing angles as for the masses, the latter in units of the light neutrino mass scale m_ν .

be around the keV scale, while the other two can easily be considerably heavier (a mass of about GeV is the lower bound to be fulfilled, but they could easily be much heavier). One important bonus of this approach is that, due to the structure of the FN charge assignments, the seesaw mechanism will be guaranteed to work if it works with all FN charges set to zero, so that we can be sure that the existence of a keV particle will not lead to any problems from that side. Furthermore, we discuss the requirements and potential problems that could arise when applying the FN mechanism in certain frameworks. Interestingly, although FN charge assignments might seem relatively arbitrary at the first sight, it is easy to find situations where the different conditions are so restrictive as to render the different sectors incompatible. This discussion leads to a systematic reduction of the arbitrariness involved, and it turns out that an ideal framework for our purpose is an $SU(5)$ GUT, augmented by two FN flavon fields. Due to the relatively low seesaw scale involved, typical constraints from RGE running and the corresponding LFV effects can be avoided. Finally, we show analytically that democratic structures of the mass matrices are not enough to explain all mass ratios, but once we depart from this structure only slightly, the agreement with the data improves. In order to find fully working models, which are also in agreement with the experimental constraints on the leptonic mixing angles, we finally perform a numerical analysis that perfectly justifies our argumentations given before.

Acknowledgements

We would like to thank J. Barry, J. Bergström, C. Luhn, K. L. McDonald, and H. Zhang for useful discussions, and we are especially grateful to Sheldon, Leonard, Howard, and Rajesh for keeping our motivation always at the top level. The work of AM is supported by the Royal Institute of Technology (KTH), under project no. SII-56510. VN acknowledges Research Grants funded jointly by Ministero dell'Istruzione, dell'Università e della Ricerca

(MIUR), by Università di Torino, and by Istituto Nazionale di Fisica Nucleare within the *Astroparticle Physics Project* (MIUR contract number: PRIN 2008NR3EBK; INFN grant code: FA51).

Appendix A: Stepwise diagonalization

We report in this appendix the explicit expressions of the approximately unitary mixing matrices that diagonalize the charged leptons mass matrix and the neutrino mass matrices for Assignment 1, Scenarios A and B, with both democratic and non-democratic Dirac Yukawa matrices.

Charged leptons

Here we report the mixing matrices that diagonalize the charged leptons mass matrix, as reported in Eq. (37):

$$\begin{aligned}
U_A &= \begin{pmatrix} 1 & 0 & a_1\lambda \\ 0 & 1 & 0 \\ -a_1^*\lambda & 0 & 1 \end{pmatrix}, \quad U_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b_1\lambda^2 \\ 0 & -b_1^*\lambda^2 & 1 \end{pmatrix}, \quad U_C = \begin{pmatrix} 1 & 0 & c_1\lambda^3 \\ 0 & 1 & 0 \\ -c_1^*\lambda^3 & 0 & 1 \end{pmatrix}, \\
U_D &= \begin{pmatrix} 1 & \lambda & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e_1\lambda^4 \\ 0 & -e_1^*\lambda^4 & 1 \end{pmatrix}, \quad U_F = \begin{pmatrix} 1 & 0 & f_1\lambda^5 \\ 0 & 1 & 0 \\ -f_1^*\lambda^5 & 0 & 1 \end{pmatrix}, \\
U_G &= \begin{pmatrix} 1 & g_1\lambda^3 & 0 \\ -g_1^*\lambda^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A-1}
\end{aligned}$$

with

$$\begin{aligned}
a_1 = b_1^* = c_1 &= R_0 e^{-i\alpha_0}, \\
e_1 &= R_0 e^{i\alpha_0} (R_0^2 e^{-2i\alpha_0} - 2R_0^2 + e^{-2i\alpha_0} + 2), \\
f_1 &= -R_0^3 e^{-i\alpha_0}, \\
g_1 &= -R_0^2 (R_0^2 + e^{-2i\alpha_0}). \tag{A-2}
\end{aligned}$$

Light neutrinos, Assignment 1, Scenario A, democratic

Here we report the mixing matrices that diagonalize the light neutrino mass matrix, see Eq. (39), for Assignment 1, Scenario A, and democratic Yukawa matrices:

$$\begin{aligned}
U_\alpha &= \begin{pmatrix} 1 & 0 & \alpha_1 \lambda \\ 0 & 1 & 0 \\ -\alpha_1^* \lambda & 0 & 1 \end{pmatrix}, \quad U_\beta = \begin{pmatrix} 1 & \lambda & 0 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma_1 \lambda^2 \\ 0 & -\gamma_1^* \lambda^2 & 1 \end{pmatrix}, \\
U_\delta &= \begin{pmatrix} 1 & 0 & \delta_1 \lambda^3 \\ 0 & 1 & 0 \\ -\delta_1^* \lambda^3 & 0 & 1 \end{pmatrix}, \quad U_\epsilon = \begin{pmatrix} 1 & \epsilon_1 \lambda^3 & 0 \\ -\epsilon_1^* \lambda^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_\zeta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\zeta_1}{\sqrt{R_0^2 + \zeta_1^2}} & -\frac{\zeta_1}{\sqrt{R_0^2 + \zeta_1^2}} \\ 0 & \frac{1}{\sqrt{R_0^2 + \zeta_1^2}} & \frac{1}{\sqrt{R_0^2 + \zeta_1^2}} \end{pmatrix}, \\
U_\eta &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{A-3}
\end{aligned}$$

with

$$\begin{aligned}
\alpha_1 &= R_0 e^{-i\alpha_0}, \\
\gamma_1 &= \frac{i}{2} R_0 \sin \alpha_0, \\
\delta_1 &= -\frac{e^{-i\alpha_0} R_0^3}{2(1 + e^{-2i\alpha_0} R_0^2)}, \\
\epsilon_1 &= -\frac{e^{-2i\alpha_0} R_0^2}{2(1 + e^{-2i\alpha_0} R_0^2)}, \\
\zeta_1 &= \left(1 - R_0^2 + \sqrt{(-1 + R_0^2)^2 + R_0^2 \cos^2 \alpha_0} \right) \sec \alpha_0. \tag{A-4}
\end{aligned}$$

Light neutrinos, Assignment 1, Scenario B, democratic

Here we report the mixing matrices that diagonalize the light neutrino mass matrix, see Eq. (42), for Assignment 1, Scenario B, and democratic Yukawa matrices:

$$\begin{aligned}
U'_\alpha &= \begin{pmatrix} 1 & 0 & \alpha'_1 \lambda \\ 0 & 1 & 0 \\ -(\alpha'_1)^* \lambda & 0 & 1 \end{pmatrix}, \quad U'_\beta = \begin{pmatrix} 1 & \beta'_1 \lambda & 0 \\ -(\beta'_1)^* \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U'_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & i\gamma'_1 \lambda^2 \\ 0 & -i(\gamma'_1)^* \lambda^2 & i \end{pmatrix}, \\
U'_\delta &= \begin{pmatrix} 1 & 0 & \delta'_1 \lambda^3 \\ 0 & 1 & 0 \\ -(\delta'_1)^* \lambda^3 & 0 & 1 \end{pmatrix}, \quad U'_\epsilon = \begin{pmatrix} 1 & \epsilon'_1 \lambda^3 & 0 \\ -(\epsilon'_1)^* \lambda^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U'_\zeta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{A-5}
\end{aligned}$$

with

$$\begin{aligned}
\alpha'_1 &= R_0 e^{-i\alpha_0}, \\
\beta'_1 &= \frac{R_0^2 - e^{2i\alpha_0}}{2R_0^2}, \\
\text{Re}(\gamma'_1) &= -\frac{R_0 \cos \alpha_0 [1 + 2R_0^2 + 3R_0^4 - 6R_0^2 \cos(2\alpha_0)]}{-1 + 3R_0^4 + 2R_0^2 \cos(2\alpha_0)}, \\
\text{Im}(\gamma'_1) &= \frac{R_0 \sin \alpha_0 [-1 - 2R_0^2 + R_0^4 - 2R_0^2 \cos(2\alpha_0)]}{-1 + 3R_0^4 + 2R_0^2 \cos(2\alpha_0)}, \\
\delta'_1 &= \frac{ie^{i\alpha_0}(e^{2i\alpha_0} + R_0^2)}{2R_0}, \\
\epsilon'_1 &= \frac{i(-e^{2i\alpha_0} + R_0^2)[-1 + R_0^4 + 2iR_0^2 \sin(2\alpha_0)]}{8R_0^6}. \tag{A-6}
\end{aligned}$$

Light neutrino, Assignment 1, Scenario A, non-democratic

Here we report the mixing matrices that diagonalize the light neutrino mass matrix, see Eq. (49), for Assignment 1, Scenario A, and non-democratic Yukawa matrices:

$$\begin{aligned}
U_\alpha &= \begin{pmatrix} 1 & 0 & \alpha_1 \lambda \\ 0 & 1 & 0 \\ -\alpha_1^* \lambda & 0 & 1 \end{pmatrix}, \quad U_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & i\beta_1 \lambda^2 \\ 0 & -i\beta_1^* \lambda^2 & i \end{pmatrix}, \quad U_\gamma = \begin{pmatrix} 1 & \frac{i\lambda}{\delta_0} & 0 \\ \frac{i\lambda}{\delta_0} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
U_\delta &= \begin{pmatrix} 1 & 0 & \delta_1 \lambda^3 \\ 0 & 1 & 0 \\ -\delta_1^* \lambda^3 & 0 & 1 \end{pmatrix}, \quad U_\epsilon = \begin{pmatrix} 1 & \epsilon_1 \lambda^3 & 0 \\ -\epsilon_1^* \lambda^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A-7}
\end{aligned}$$

with

$$\begin{aligned}
\alpha_1 &= R_0 e^{-i\alpha_0}, \\
\text{Re}(\beta_1) &= -\frac{R_0 \delta_0 \cos \alpha_0}{-1 + \delta^2}, \\
\text{Im}(\beta_1) &= \frac{R_0 \delta_0 \sin \alpha_0}{1 + \delta^2}, \\
\delta_1 &= -\frac{ie^{-i\alpha_0} R_0 (2e^{4i\alpha_0} + 2R_0^2 \delta_0^2 + e^{2i\alpha_0} [2\delta_0^2 + R_0^2 (1 + \delta_0^4)])}{2(e^{2i\alpha_0} + R_0^2)(-1 + \delta_0^4)}, \\
\epsilon_1 &= \frac{ie^{2i\alpha_0} R_0^2}{2\delta_0^3 (1 + e^{2i\alpha_0} R_0^2)}. \tag{A-8}
\end{aligned}$$

Light neutrinos, Assignment 1, Scenario B, non-democratic

Here we report the mixing matrices that diagonalize the light neutrino mass matrix, see Eq. (51), for Assignment 1, Scenario B, and non-democratic Yukawa matrices:

$$\begin{aligned}
 U'_\alpha &= \begin{pmatrix} 1 & 0 & \alpha'_1 \lambda \\ 0 & 1 & 0 \\ -(\alpha'_1)^* \lambda & 0 & 1 \end{pmatrix}, \quad U'_\beta = \begin{pmatrix} 1 & \beta'_1 \lambda & 0 \\ -(\beta'_1)^* \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U'_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & i \gamma'_1 \lambda^2 \\ 0 & -i(\gamma'_1)^* \lambda^2 & i \end{pmatrix}, \\
 U'_\delta &= \begin{pmatrix} 1 & 0 & \delta'_1 \lambda^3 \\ 0 & 1 & 0 \\ -(\delta'_1)^* \lambda^3 & 0 & 1 \end{pmatrix}, \quad U'_\epsilon = \begin{pmatrix} 1 & \epsilon'_1 \lambda^3 & 0 \\ -(\epsilon'_1)^* \lambda^3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{A-9}
 \end{aligned}$$

with

$$\begin{aligned}
 \alpha'_1 &= R_0 e^{-i\alpha_0}, \\
 \beta'_1 &= \frac{1}{2\delta_0} - \frac{e^{2i\alpha_0}}{2\delta_0 R_0^2}, \\
 \text{Re}(\gamma'_1) &= \frac{R_0 \delta_0 (-[1 - R_0^2 + R_0^4(1 + 2\delta_0^2)] \cos \alpha_0 + R_0^2(1 + 2\delta_0^2) \cos(3\alpha_0))}{-1 + R_0^4(-1 + 4\delta_0^4) + 2R_0^2 \cos(2\alpha_0)}, \\
 \text{Im}(\gamma'_1) &= \frac{R_0 \delta_0 (-[1 - R_0^2 + R_0^4(1 + 2\delta_0^2)] \sin \alpha_0 + R_0^2(1 - 2\delta_0^2) \sin(3\alpha_0))}{-1 + R_0^4(-1 + 4\delta_0^4) + 2R_0^2 \cos(2\alpha_0)}, \\
 \delta'_1 &= \frac{ie^{3i\alpha_0}(1 + e^{-2i\alpha_0} R_0^2)}{2R_0}, \\
 \epsilon'_1 &= \frac{i(R_0^6 + e^{2i\alpha_0} - R_0^4 e^{-2i\alpha_0} - R_0^2 e^{4i\alpha_0})}{8R_0^6 \delta_0^3}. \tag{A-10}
 \end{aligned}$$

References

- [1] F. Zwicky, *Helv. Phys. Acta* **6**, 110 (1933).
- [2] E. Komatsu *et al.* (WMAP), *Astrophys. J. Suppl.* **192**, 18 (2011), 1001.4538.
- [3] P. Bode, J. P. Ostriker, and N. Turok, *Astrophys. J.* **556**, 93 (2001), astro-ph/0010389.
- [4] H. J. de Vega, P. Salucci, and N. G. Sanchez (2010), 1004.1908.
- [5] H. J. de Vega and N. G. Sanchez, *Mon. Not. Roy. Astron. Soc.* **404**, 885 (2010), 0901.0922.
- [6] T. Asaka, S. Blanchet, and M. Shaposhnikov, *Phys. Lett.* **B631**, 151 (2005), hep-ph/0503065.

- [7] M. Shaposhnikov and I. Tkachev, Phys. Lett. **B639**, 414 (2006), [hep-ph/0604236](#).
- [8] F. Bezrukov, D. Gorbunov, and M. Shaposhnikov, JCAP **0906**, 029 (2009), [0812.3622](#).
- [9] F. Bezrukov and D. Gorbunov, JHEP **1005**, 010 (2010), [0912.0390](#).
- [10] F. Bezrukov, H. Hettmansperger, and M. Lindner, Phys. Rev. **D81**, 085032 (2010), [0912.4415](#).
- [11] D. Aristizabal Sierra, J. Kubo, D. Restrepo, D. Suematsu, and O. Zapata, Phys. Rev. **D79**, 013011 (2009), [0808.3340](#).
- [12] G. B. Gelmini, E. Osoba, and S. Palomares-Ruiz, Phys. Rev. **D81**, 063529 (2010), [0912.2478](#).
- [13] Y. Grossman and D. J. Robinson, JHEP **01**, 132 (2011), [1009.2781](#).
- [14] M. Shaposhnikov, Nucl. Phys. **B763**, 49 (2007), [hep-ph/0605047](#).
- [15] M. Lindner, A. Merle, and V. Niro, JCAP **1101**, 034 (2011), [1011.4950](#).
- [16] A. Kusenko, F. Takahashi, and T. T. Yanagida, Phys. Lett. **B693**, 144 (2010), [1006.1731](#).
- [17] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), [hep-ph/9905221](#).
- [18] P. Minkowski, Phys. Lett. **B67**, 421 (1977).
- [19] T. Yanagida In Proceedings of the Workshop on the Baryon Number of the Universe and Unified Theories, Tsukuba, Japan, 13-14 Feb 1979.
- [20] M. Gell-Mann, P. Ramond, and R. Slansky Print-80-0576 (CERN).
- [21] S. L. Glashow, NATO Adv. Study Inst. Ser. B Phys. **59**, 687 (1980).
- [22] R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980).
- [23] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. **B147**, 277 (1979).
- [24] W. Buchmüller and T. Yanagida, Phys. Lett. **B445**, 399 (1999), [hep-ph/9810308](#).
- [25] G. Altarelli and F. Feruglio, Phys. Lett. **B451**, 388 (1999), [hep-ph/9812475](#).
- [26] H. Kamikado, T. Shindou, and E. Takasugi (2008), [0805.1338](#).
- [27] N. Haba, R. Takahashi, M. Tanimoto, and K. Yoshioka, Phys. Rev. **D78**, 113002 (2008), [0804.4055](#).

- [28] M.-C. Chen and K. T. Mahanthappa, Int. J. Mod. Phys. **A18**, 5819 (2003), [hep-ph/0305088](#).
- [29] A. K. Datta, L. Everett, and P. Ramond, Phys. Lett. **B620**, 42 (2005), [hep-ph/0503222](#).
- [30] J. Barry, W. Rodejohann, and H. Zhang (2011), [1105.3911](#).
- [31] G. L. Kane, S. F. King, I. N. R. Peddie, and L. Velasco-Sevilla, JHEP **08**, 083 (2005), [hep-ph/0504038](#).
- [32] F. Plentinger, G. Seidl, and W. Winter, Nucl.Phys. **B791**, 60 (2008), [hep-ph/0612169](#).
- [33] F. Plentinger, G. Seidl, and W. Winter, Phys.Rev. **D76**, 113003 (2007), [0707.2379](#).
- [34] F. Plentinger, G. Seidl, and W. Winter, JHEP **0804**, 077 (2008), [0802.1718](#).
- [35] S. Niehage and W. Winter, Phys.Rev. **D78**, 013007 (2008), [0804.1546](#).
- [36] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. **B530**, 167 (2002), [hep-ph/0202074](#).
- [37] P. H. Frampton, S. T. Petcov, and W. Rodejohann, Nucl. Phys. **B687**, 31 (2004), [hep-ph/0401206](#).
- [38] S. T. Petcov and W. Rodejohann, Phys. Rev. **D71**, 073002 (2005), [hep-ph/0409135](#).
- [39] J. Sato and K. Tobe, Phys. Rev. **D63**, 116010 (2001), [hep-ph/0012333](#).
- [40] N. G. Deshpande, J. F. Gunion, B. Kayser, and F. I. Olness, Phys. Rev. **D44**, 837 (1991).
- [41] K. Choi, E. J. Chun, K. Hwang, and W. Y. Song, Phys. Rev. **D64**, 113013 (2001), [hep-ph/0107083](#).
- [42] L. Wolfenstein, Nucl. Phys. **B186**, 147 (1981).
- [43] S. Kanemura *et al.*, Eur. Phys. J. **C51**, 927 (2007), [0704.0697](#).
- [44] G. G. Ross *Grand Unified Theories*, Reading, USA: Benjamin/Cummings (1984) 497 P. (Frontiers In Physics, 60).
- [45] N. Irges, S. Lavignac, and P. Ramond, Phys. Rev. **D58**, 035003 (1998), [hep-ph/9802334](#).
- [46] L. E. Ibanez and G. G. Ross, Phys. Lett. **B332**, 100 (1994), [hep-ph/9403338](#).
- [47] V. Jain and R. Shrock, Phys. Lett. **B352**, 83 (1995), [hep-ph/9412367](#).

- [48] T. Asaka, Phys. Lett. **B562**, 291 (2003), [hep-ph/0304124](#).
- [49] T. Schwetz, M. Tórtola, and J. W. F. Valle (2011), [1103.0734](#).
- [50] S. Antusch, J. Kersten, M. Lindner, M. Ratz, and M. A. Schmidt, JHEP **03**, 024 (2005), [hep-ph/0501272](#).
- [51] J.-W. Mei, Phys. Rev. **D71**, 073012 (2005), [hep-ph/0502015](#).
- [52] J. Bergström, M. Malinský, T. Ohlsson, and H. Zhang, Phys. Rev. **D81**, 116006 (2010), [1004.4628](#).
- [53] B. W. Lee and R. E. Shrock, Phys. Rev. **D16**, 1444 (1977).
- [54] A. Blum and A. Merle, Phys. Rev. **D77**, 076005 (2008), [0709.3294](#).
- [55] S. P. Martin (1997), [hep-ph/9709356](#).
- [56] J. Hisano, T. Moroi, K. Tobe, and M. Yamaguchi, Phys. Rev. **D53**, 2442 (1996), [hep-ph/9510309](#).
- [57] M. L. Brooks *et al.* (MEGA), Phys. Rev. Lett. **83**, 1521 (1999), [hep-ex/9905013](#).
- [58] M. Ahmed *et al.* (MEGA), Phys. Rev. **D65**, 112002 (2002), [hep-ex/0111030](#).
- [59] H. Natori (MEG), Nucl. Phys. Proc. Suppl. **210-211**, 241 (2011).
- [60] M. Raidal *et al.*, Eur. Phys. J. **C57**, 13 (2008), [0801.1826](#).
- [61] J. Kopp, M. Lindner, V. Niro, and T. E. Underwood, Phys.Rev. **D81**, 025008 (2010), [0909.2653](#).
- [62] H. Murayama and A. Pierce, Phys.Rev. **D65**, 055009 (2002), [hep-ph/0108104](#).
- [63] K. Kobayashi *et al.* (Super-Kamiokande), Phys. Rev. **D72**, 052007 (2005), [hep-ex/0502026](#).
- [64] H. Nishino *et al.* (Super-Kamiokande), Phys. Rev. Lett. **102**, 141801 (2009), [0903.0676](#).
- [65] L. Randall and C. Csaki (1995), [hep-ph/9508208](#).
- [66] Y. Nomura, D. Tucker-Smith, and N. Weiner, Nucl. Phys. **B613**, 147 (2001), [hep-ph/0104041](#).
- [67] Y. Kawamura, Prog.Theor.Phys. **105**, 999 (2001), [hep-ph/0012125](#).
- [68] L. J. Hall and Y. Nomura, Phys.Rev. **D66**, 075004 (2002), [hep-ph/0205067](#).

- [69] J. R. Ellis, J. L. Lopez, and D. V. Nanopoulos, Phys.Lett. **B371**, 65 (1996), [hep-ph/9510246](#).
- [70] H. Murayama and D. Kaplan, Phys.Lett. **B336**, 221 (1994), [hep-ph/9406423](#).
- [71] Y. Kawamura (2009), [0906.3773](#).
- [72] K. Babu and S. M. Barr, Phys.Lett. **B381**, 137 (1996), [hep-ph/9506261](#).
- [73] S. Antusch, J. Kersten, M. Lindner, M. Ratz, and M. A. Schmidt, *Mixing Parameter Tools (MPT)*, <http://users.physik.tu-muenchen.de/rge/MPT/index.html>.
- [74] J. Sato and T. Yanagida, Phys. Lett. **B493**, 356 (2000), [hep-ph/0009205](#).